

Linear Programming: Solid New Algorithm

Many people think that abstract mathematics is a nightmare. In truth, it is a special sort of dream. Both mathematics and dreams are often bizarre and hypothetical products of human imagination; both are exotic and intangible extensions of human experience, yet both frequently provide fertile and ingenious metaphors for reality.

Who would have dreamed that speculations about parallel lines that intersect, or about straight lines that return from infinity to their point of origin — the dream world of non-Euclidean geometry — would offer a better model for the geometry of the universe than the perceived reality of classical Euclidean geometry? Equally astonishing — perhaps even ironic — is the recent realization that the “imaginary” numbers that medieval mathematicians conjured up to solve “unsolvable” equations such as $x^2 = -1$ are better suited to describe the fundamental particles of matter than are the so-called “real” numbers of antiquity.

That mathematical imaginings can influence reality is what Princeton physicist Eugene Wigner once called the “unreasonable effectiveness” of mathematics: Even though mathematics itself is a rational discipline there is no apparent rational basis for its uncanny ability to mimic or model reality. It is, after all, a special kind of dream.

The most recent example of a mathematical dream being harnessed to solve a real-world problem is the development of a new method of solving problems arising out of the application of game theory to economics and the management of organizations, the so-called linear programming problems. It renders more surely soluble a class of problems that were sometimes not practically calculable and holds out the possibility that a similar vision may provide insight into a totally insoluble class, those whose complexities grow with exponential speed.

The possibility of a mathematical theory of games was first suggested in 1928 by the late John von Neumann. Sixteen years later, in 1944, von Neumann and the economist Oskar Morgenstern published a thick treatise on this unrealized mathematical vision, arguing that the hypothetical world of games provided an effective metaphor for economic reality. Their book, *The Theory of Games and Economic Behavior*, has since had such a profound influence on the conduct of human affairs

Solving problems in the
applications of game theory
is an uphill struggle

BY LYNN ARTHUR STEEN

that it has frequently been compared in this regard to Marx's *Das Kapital*.

The theory of games is concerned with finding the best strategy to achieve some objective. It has been used (and perhaps abused) in political strategy, foreign policy, war-games, and in controlling the national economy. Like almost any other mathematical theory, it has spawned variations that have now become full-grown research fields: linear and non-linear programming, operations research, theory of optimization, cybernetics and control theory.

von Neumann and Morgenstern's work appeared near the end of World War II, at a time when the industrial and scientific leaders of the United States were acutely aware of the immense difficulty of scheduling in an optimum way the quantity and timing of activities in a complex organization. The war effort did more than just strain the industrial capacity of the United States: Like the current world energy crisis, it imposed urgent requirements of efficiency on an unprecedented, massive scale. It strained our ability to plan as much as our ability to produce.

Consider, to take a trivial example, the task of assigning 50 differently qualified people to 50 different jobs in such a way that the most work gets done. There are more than 10^{30} different ways that these job assignments can be made. To check out the effect of each, even at the unimaginable rate of 1,000,000 per second, would take 10^{35} centuries — longer than the estimated life of the universe. (This may help explain why large bureaucracies have such a hard time operating with any semblance of efficiency.)

In 1948 George Dantzig, currently at Stanford University, devised an effective, systematic procedure for finding the optimal solution in problems of this type. Dantzig's method is based on an apprehension of a fantastic “hyper” polyhedron in a geometric space with as many dimensions as there are constraints in the original problem. Here is Dantzig's dream:

Each requirement of the original problem (for example, that the total number of

workers assigned to the job not exceed the supply of available workers) determines a surface — usually a generalized plane, or what mathematicians call a “hyperplane” — in a space with a very large number of dimensions. Points in this space represent possible plans of action (a certain number of workers, a certain supply of raw materials). Each surface (or hyperplane) divides this enormous space into two regions: On one side of the surface are points corresponding to feasible plans of action, while on the other side are points that are infeasible because they violate the requirement that the surface represents.

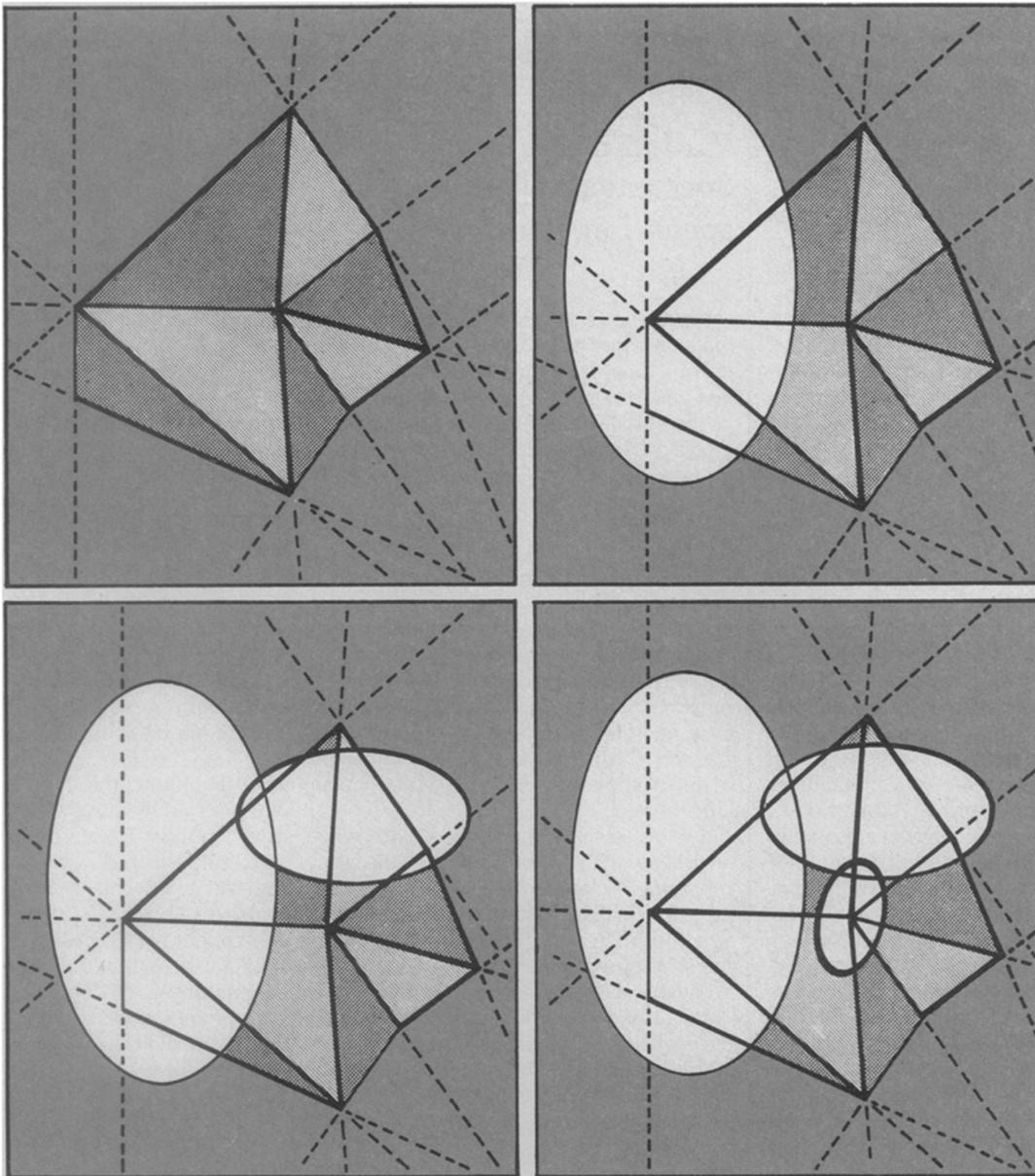
In a typical large industrial problem there may be several dozen requirements, perhaps even a few hundred. To each requirement there corresponds a hypersurface, and these dozen (or hundreds of) surfaces intersect in a dazzling, incredible display of intertwined curves and surfaces in a Euclidean space of perhaps 50 or 100 dimensions. The planner's job is to find in this phantasmagoric haystack of hypersurfaces the one point that represents a plan of action where everything works in an optimal fashion.

Dantzig's solution is to move from corner to corner on the surface of the hyper-polyhedron formed by the hyperplanes, at each corner selecting the route that most rapidly approaches the objective of maximum profit or minimum cost. This procedure is now commonly (perhaps ironically) known as the “simplex algorithm” because mathematicians call the region of space bounded by all the hypersurfaces a “simplex.”

Because the equations of the hyperplanes are linear equations, the problem that the simplex algorithm solves is commonly known as linear programming (or often as just “LP”). In recent years about one-fourth of all computer time devoted to scientific computation has been consumed by solutions to linear programming problems and their many variations. With computer time selling for thousands of dollars per hour, the speed of the simplex algorithm has enormous economic as well as mathematical consequences.

The simplex algorithm climbs incessantly uphill, always gaining on its objective as it searches for the best point. While this single-mindedness ensures that a solution will be achieved, it does so at the price of guaranteed efficiency. As the algorithm moves from corner to corner, it invariably overlooks opportunities for

Some mathematical problems once considered almost incalculably difficult to solve, such as those involving selecting the optimum arrangement of numerous variables, have met with some success through algorithms in linear programming — but there has still been room for improvement. **THE OLD WAY:** Devised in 1948, this method (top left) essentially involves moving from corner to corner of a complex figure formed by planes intersecting in a many-dimensional



space. Each plane represents a different requirement of the problem, with the solver at each corner selecting the route that most rapidly approaches the objective. But extremely complex problems still take time because it is only the next corner, rather than the final solution, that is clearly in view. **THE NEW WAY:** Here a sequence of many-dimensional ellipsoids are "constructed" to slide sideways, constrained by the planes, so that their centers are seen to gradually converge on the solution.

sacrifice, in which a digression from the objective may be worthwhile if it leads to a greater subsequent gain.

It would be nice if the simplex algorithm were as smart as the mountain climber who will descend into a valley in order to reach a new slope on which he can better launch his assault on the summit. But to do this, the mountain climber must have the advantage of foresight, provided either by a map or a guide, or by vision from a high vantage point. As there are no clues in the local terrain that could reveal to the climber the advantages of sacrifice, so are there none for Dantzig's simplex algorithm. It does the best it can with limited information, plodding slowly but incessantly toward its objective, even if more direct routes are possible.

Despite this handicap, the simplex algorithm is remarkably effective: For most large problems the route it picks is almost as fast as the best route, and it usually finds the solution in a reasonable amount of time. But sometimes the simplex al-

gorithm can take forever. (Not literally, of course, since it must quit sometime. But 10^{40} or even 10^{20} seconds is as good as forever insofar as human lifetime is concerned, since there are fewer than 10^{16} seconds in a century.) It can, seemingly, get so lost that it never approaches its objective.

In 1972 Victor Klee and George Minty discovered a fog of combinatorial possibilities so thick that the simplex algorithm gets totally lost: It wanders and wanders from corner to corner, always going uphill, but making very little progress toward the summit. Klee and Minty's discovery showed that linear programming could be almost as intractable as the infamous traveling salesman problem, or others of the so-called NP complete class (SN: 5/8/76, p. 298). These are problems whose complexity appears to grow exponentially as their size increases. Exponential growth is so dramatic that it can convert trivial problems into unsolvable ones in the blink of an eye.

Polynomial growth is more manageable: If the time required to solve a problem increases as the square or the cube (or even as the 5th or 6th power) of the problem's size, then a modern computer with efficient programs can cope with the extra burden when the problem increases in size. For example, if a problem with 20 parameters takes about a second of computer time to solve with an n^3 algorithm — one that increases as the cube of the size of the problem, n being the number of parameters — then that same problem with 50 parameters will take about 12 seconds. But if the algorithm were to grow exponentially, say like 3^n , then the one-second solution would become a several-thousand-century nightmare when the size of the problem increased from 20 to 50.

These fundamental facts of combinatorial life divide the world of problems into two major classes — those with polynomial time algorithms, and those without them. The former problems are solvable;

the latter, for all practical purposes, unsolvable. For many years linear programming seemed to hover in between: Although the simplex algorithm usually worked in polynomial time, sometimes it did not. Theorists were able to prove that linear programming was not as bad as the terrible NP complete problems, so they thought that it represented a new class of problems that was in between the easy (polynomial) and hard (NP complete) problems.

Now, suddenly, all that has changed. Several months ago the Russian mathematician L. G. Khachian published in *DOKLADY*, the Proceedings of the Academy of Science of the USSR, the outline of a totally new algorithm for linear programming that is guaranteed to finish in polynomial time. Just last month Khachian's algorithm was confirmed and elaborated by Peter Gacs and Laszlo Lovasz, two Hungarian mathematicians currently visiting in the United States.

As might be expected, Khachian's algorithm is totally different from any heretofore proposed. Instead of following the edges of the simplex or tracing the branches of a large decision tree — as many other combinatorial algorithms do — Khachian exploited classical Euclidean geometry inflated into high dimensions.

The key idea is to construct a sequence of high dimensional ellipsoids that slide sideways, under the influence of the constraining surfaces, so their centers gradually converge on the solution. The volumes of the ellipsoids gradually decrease, thus ensuring that the solution will be located within a reasonable number of steps. When written in formulas, Khachian's algorithm is sufficiently simple that it can be programmed on a hand calculator. It is certainly no harder to use than is the simplex algorithm, and may, when fully understood, be much simpler.

So despite a quarter-century of successful use of the simplex algorithm, it turns out that it is really not the only way, nor even perhaps the best way to solve linear programming problems. Moreover, Khachian's scheme raised the tantalizing possibility that if linear programming, which was nearly NP complete, could be reduced to polynomial time, then perhaps a similar stroke of genius could break through the NP barrier with a polynomial time solution to these problems as well.

What matters most is that it is a totally new idea, a refreshing dream that reveals how geometry and algebra can interact to their mutual benefit. Khachian's method appears to exhibit greater "intelligence" than Dantzig's simplex method, because it is able to sacrifice in order to assure greater future gains. Nobody yet fully understands how it is able to do this: As we are unable to explain the origins of dreams or ideas, so we are yet unable to comprehend the way in which Khachian's moving ellipsoids can behave as if they were intelligent. □

OFF THE BEAT

Pardon me, does anyone here speak English?

Mathematics is a kind of language. To really get into it you have to be able to think in those funny symbols and the ideas they relate to, nonverbally if possible. That is one of the reasons it is often difficult to interpret mathematical sciences for a general public. American schools have seldom equipped people to think like that. Mostly people have learned it for themselves.

Similarly, American schools have rarely equipped people to think in any of the natural verbal languages of humankind except the English native to the country's majority. Great Britain and the Queen's other dominions are largely in similar case, and together that makes a large reason why English is the accepted language of international science. Accepted by a kind of default.

Some years ago Charles de Gaulle, who always gave the impression that he was refighting the battle of Bouvines, ordered French scientists to deliver their communications in French. For a while they complied. But they wanted audiences, and so gradually they drifted back to English. I once attended an international conference where simultaneous translation was provided à la United Nations. People found the earphones uncomfortable, and waiting for the interpreters inhibited the spontaneity of the question-and-answer portions. Those who could, began to speak English without waiting for the translators.

This situation has been accepted quite smugly for a long time, but recently a couple of letters in scientific publications have asked how much of these "international" proceedings are really understood by people whose native language is not English. As I sat through international meetings this summer — all proceedings of which were in English—I wondered too. On the one hand there were speeches with reference to baseball or American football or wisecracks based on English idioms and proverbs. It's all right to say that these were not the scientific parts of the talk (sometimes they were), but the foreigner whose English is imperfect doesn't know that. On the other hand, there were extremely painful attempts to communicate in an English that was obviously a very foreign language.

Montreal is a good place to watch these things. An international conference in Montreal is certain to be dominated numerically by English-speaking North Americans, yet in Montreal they are

slightly off their base. One should not exaggerate the exotic quality of Montreal. As the man in the *brasserie* said, it's a North American city. In the corridors of the University of Montreal were posters advertising a *Ligue de Balle-Molle* (softball league). A Frenchman would have required an explanation of that phrase.

Yet the city looks and sounds more French all the time. If you can't read French, you won't know where you're not allowed to park, nor where they tow it if you do. The atmosphere seemed to inhibit some people. There is a small bank branch in the University building where the sessions were held. Few of the 3,000 astronomers present seemed to use it, although they made long lines at a currency exchange near the registration desk. Was it that they could not speak French to the tellers? Did they not know that this, like all Montreal banks, would gladly buy U.S. currency? Were they reluctant to ask?

Finally, a story about some Germans in a restaurant in the West End of Montreal. One of them spoke a little French and got from the waitress descriptions of the dishes on the menu, which she then related to her companions in German. Later they were wondering if they would get what they thought they had ordered. One of them said plaintively, "But I speak English." I seldom interfere at tables not my own, but I wanted to go over to him and say, "Then talk to her in English, man." The waitress's better language was obviously French, but her English was adequate.

Linguistic inhibitions are thus not solely an anglophone problem, but it is the anglophones who are the loudest group in international science today, and it is they who will have to do some reaching out. English is likely to remain the most widely used language because of the number of scientists whose native language is English and because those from Asia, Africa and South America are often educated in English. But perhaps anglophone scientists could learn foreign languages more — if only to be familiar with the difficulties of a foreigner listening to English or to be able to ask and answer questions in languages more comfortable for non-English colleagues. Provision for delivery of papers in non-English languages could be made again, with translators on duty if needed. It might improve communications.

But it won't be perfection. Even when everybody understands the information conveyed, there can be problems. I was standing in a bar in Montreal one night when I heard a voice beside me say, "*As-tu l'heure?*" (Have you the time?)

I looked at my watch and replied, "*Midi trente.*" (Half past noon.)

When the giggling started, I realized what I had said and corrected it: "*Minuit trente.*" (Half past midnight.)

The mind has to be in gear as well as the vocabulary.

—Dietrick E. Thomsen