New Models of the Real-Number Line

Recent developments in mathematical logic reveal that there are a number of alternative ways of defining the continuum, or connected number system, to include all the real numbers

by Lynn Arthur Steen

Virtually all of mathematics and much of science is based on the abstract concept of the real-number line: the continuum, or connected number system, that includes all the real numbers—zero, positive and negative integers, rational numbers (fractions) and irrational numbers (such as π)—but that excludes the "unreal," or complex, numbers—expressions containing the imaginary number $\sqrt{-1}$. The real-number continuum not only provides the natural setting for all the operations of arithmetic and calculus but also serves as our only intellectual model of time and (one-dimensional) space. The properties of the continuum were organized into a coherent axiomatic framework during the 19th century and have been accepted and promulgated with great conviction by most contemporary mathematicians.

Yet in spite of the present unanimity of opinion concerning the exact structure of the continuum several significant alternative systems have been discovered during the past 10 years. Although none of these new models reflects any logical flaw in the 19th-century theory, their very existence shows quite clearly that the epistemological foundation of mathematical analysis is far from settled.

Although mathematics is sometimes called a science, it is usually distinguished from science by its relative independence from empirical considerations. The intellectual models of science are judged by their ability to explain the observed properties of the universe, whereas those of mathematics are judged (by mathematicians) according to their consistency and beauty and (by scientists) according to their utility. Plato's description of mathematics as the discovery of the properties of objects in an ideal universe—the universe of Platonic ideals—has been the most enduring and popular philosophy of mathematics, since it provides mathematics with the (scientific) discipline of conforming to some kind of perceived reality together with the freedom to escape the bonds of empiricism.

Most working mathematicians, at least those not directly engaged in problems of mathematical logic, tend to be Platonists in the sense that they talk and act as if the abstract objects they study have some kind of enduring ideal existence. Standing in extreme opposition to the Platonists are the Formalists, who maintain that the essence of mathematics is not in its meaning but in its form.

The problem of the nature of the real-number line is viewed quite differently by adherents to these two schools of thought. A Formalist is likely to say that the real-number line is whatever we define it to be; if we have several competing definitions, then we will have several different real-number lines and mathematics will be enriched by their presence. A Platonist, on the other hand, would be inclined to wonder which of the different models represented the "real" real-number abstraction of the space and time continuum.

Like beauty, mathematics exists in the eye of the beholder. Unlike beauty, however, mathematics enjoys an unparalleled worldwide reputation for objectivity. If you labor to discover the properties of the real-number continuum and if I do likewise, we shall reach the same conclusions. Science has a similar objectivity. But whereas the objectivity of science is a rather plausible consequence of its conformity with reality, the objectivity of mathematics is harder to explain. Why should the ideal mathematical universe in your mind be the same as the one in my mind?

Two centuries ago Immanuel Kant attempted to answer this question by postulating the a priori existence in the human mind of a kernel of intuitive mathematical and geometrical truth. The considerable influence of Kant's philosophy reinforced the widely held view that the axioms of mathematics should be self-evident truths. Euclidean geometry became the archetype of mathematics, since it was a beautiful, useful theory created by logical deduction from certain (nearly) self-evident axioms. Indeed, Kant himself had asserted that the geometrical intuition of Euclidean space and of the time continuum was one of the a priori characteristics of the human mind.

Thus it came as a considerable shock to the intellectual community of the early 19th century to learn of the discovery of non-Euclidean geometries, in which one of Euclid's axioms (the parallel postulate) did not hold. One significant consequence of this discovery was that philosophers and mathematicians began to regard axioms not as self-evident truths but rather as arbitrary rules, subject only to the requirement of consistency. For the axioms to be great the theory derived from them had to be both beautiful and useful, but to be mathematics the theory had only to be consistent.

Yet in spite of the widespread 20th-century consensus on consistency as the sole criterion of mathematical truth, the vast majority of mathematicians and scientists maintain a Platonic view of the continuum. Nearly everyone who studied calculus in the past 50 years was expected either to have a clear intuitive (Platonic) image of the real-number line or to believe that all its properties were consequences of some 10 to 15 supposedly self-evident truths that describe what is called formally a "complete ordered field" and informally the "real-number system" [*see illustration on this page*].

A cynic might view this program as indoctrination in Platonism, since its clear purpose is to convince students of something their professors have accepted, namely that there is a unique real-number line with certain self-evident properties, and that mathematicians have succeeded, by listing the axioms of a complete ordered field, in capturing the essence of this line in a dozen or so sentences. Thus mathematicians now talk about *the* real-number line just as mathematicians and philosophers of the 18th century talked about *the* geometry.

Of course, the existence of different geometries does not necessitate the existence of different real-number lines. Nonetheless, in 1931 Kurt Gödel showed that in any mathematical system sufficiently large to contain arithmetic, there will always be undecidable sentences: statements about the system that can be neither proved nor disproved by logical deduction from the axioms [see "Gödel's Proof," by Ernest Nagel and James R. Newman; SCIENTIFIC AMERICAN, June, 1956]. Gödel's now famous "undecidability theorem" implies that in the Platonic universe of ideal mathematical objects there are many—in fact, infinitely many—objects that satisfy the axioms for the real-number line, since each undecidable proposition about the real-number line may be true in one ideal model and false in another.

In 1963, more than 30 years after Gödel proved that different models for the real-number axioms must exist, Paul J. Cohen of Stanford University actually constructed some models in which Georg Cantor's famous "continuum hypothesis" was false [see "Non-Cantorian Set Theory," by Paul J. Cohen and Reuben Hersh; SCIENTIFIC AMERICAN, December, 1967]. Cohen's methods have been applied extensively over the past eight years to yield a large variety of alternative models in all areas of mathematics. Later in this article I shall show how to construct one of these models for the real-number line, but first I should like to discuss an entirely different source of alternative models, since Gödel's undecidability theorem is not the only line of attack on the Platonic ideal of the real-number line.

Ever since Isaac Newton and Gottfried Leibniz laid the foundations of calculus in the late 17th century, mathematicians, philosophers and physicists have been quarreling about whether or not the real-number line contains infinitely small objects called infinitesimals. Infinitesimals played a key role in the development of the definitions and notations of calculus. Indeed, during the 18th and 19th centuries the name for what we now call simply "calculus" was "calculus of infinitesimals." What Newton and Leibniz did was to show how one could calculate with infinitesimals and obtain reliable results, results that moreover could not be obtained by any other method.

The calculus of infinitesimals was received with profound skepticism by many philosophers who, following Aristotle, abhorred the absolute infinite. In his *Physics* Aristotle distinguished between the potential infinite and the absolute infinite, accepting the former but rejecting the latter as untenable, or beyond the firm grasp of the human mind. (In taking this position Aristotle was merely reflecting the widespread Greek mistrust of the infinite, most popularly illustrated by the paradoxes of Zeno.) The philosopher Leibniz, somewhat chagrined at what the mathematician Leibniz had wrought, squirmed out of the dilemma by describing infinitesimals as "fictions, but useful fictions." Meanwhile Bishop Berkeley scorned Newton's infinitesimals (or fluxions, as Newton called them) as "ghosts of departed quantities."

But mathematics flourished in spite of the philosophers, awakened by the calculus of infinitesimals as it had not been since the glories of Athens and Alexandria. The Platonic image of the real-number line was as yet only a vague ideal, and calculus developed more as a descriptive science than as a deductive logical system. It was not until the 19th century that mathematicians began to echo the philosophers' skepticism; it was only then, as the axioms for the real-number system were distilled from the great unorganized mass of mystical properties, that it became clear that the existence of infinitesimals was inconsistent with these axioms.

This inconsistency is an immediate consequence of one of the most char-

1. **Addition and Multiplication**
   If x and y are real numbers, then so are $x + y$ and xy.

2. **Associativity**
   If w, x and y are real numbers, then $(w + x) + y = w + (x + y)$ and $(wx)y = w(xy)$.

3. **Commutativity**
   If x and y are real numbers, then $x + y = y + x$ and $xy = yx$.

4. **Distributivity**
   If w, x and y are real numbers, then $w(x + y) = wx + wy$.

5. **Identities**
   There exist two special numbers z (or 0) and u (or 1) called the zero and the unit that satisfy $x + z = x$ and $xu = x$ for all real numbers x.

6. **Additive Inverse**
   If x is any real number, there is another real number denoted by $-x$ and called the negative or additive inverse of x that satisfies $x + -x = z$, where z is the zero.

7. **Multiplicative Inverse**
   If x is any real number except zero, there is another real number x^{-1}, called the reciprocal or multiplicative inverse of x, that satisfies $x x^{-1} = u$, where u is the unit.

8. **Trichotomy**
   If x and y are real numbers, then either $x < y$ or $x = y$ or $x > y$.

9. **Transitivity**
   If $w < x$ and $x < y$, then $w < y$.

10. **Isotony**
    If $x < y$, then $x + w < y + w$; if $x < y$ and $w > z$ (where z is zero), then $xw < yw$.

11. **Completion**
    Suppose E is a set of real numbers that has an upper bound, that is, suppose there is some real number x such that $y < x$ whenever y is a member of the set E. Then E has a least upper bound, that is, an upper bound x that is less than or equal to every other upper bound for E.

ELEVEN AXIOMS are customarily used to define the real-number line, which is known formally as a complete ordered field. The first seven axioms define a field; the first 10 define an ordered field. The symbol $>$ means "greater than"; the symbol $<$ means "less than."

acteristic properties of infinitesimals, namely that any multiple of an infinitesimal is still an infinitesimal. For instance, the infinitesimal dx used in calculus is smaller than every ordinary positive real number, and so is every multiple $m(dx)$ for any positive integer m. Accordingly the set M of all multiples of the infinitesimal dx has many upper bounds (any ordinary positive real number will do), but it has no least upper bound, since for any given number b that is an upper bound for M the smaller number $b - dx$ will also be an upper bound for M. Thus M fails to satisfy the final axiom for the real-number system, the "completeness" axiom.

Hence at the same time that geometers were being forced to change their criterion of truth from self-evidence to consistency, analysts were discovering that the supposedly self-evident axioms for the calculus of infinitesimals were inconsistent. What was to be done? The answer, developed principally by Augustin Cauchy and Karl Weierstrass, was to abandon the infinitesimals but keep the calculus (whence our present abbreviated name for the subject). Cauchy reformulated the foundations of calculus by substituting the concept of a limit for that of an infinitesimal; his method was a return to the Aristotelian concept of the potential infinite as the only secure basis for reasoning.

Weierstrass extended Cauchy's work by defining the concept of a limit in terms of the more primitive concept of real numbers. The Cauchy-Weierstrass approach to calculus, now widely taught, placed the epistemological foundation of calculus squarely on the shoulders of the real-number line. Although many users of calculus continue to prefer the intuitive language of infinitesimals, virtually all 20th-century mathematicians have adopted the definitions and concepts of Cauchy and Weierstrass.

Within the past decade, however, Abraham Robinson of Yale University developed a consistent mathematical theory of infinitesimals. This new theory, called "nonstandard analysis," resuscitated the discredited ideas of the actual infinite and actual infinitesimal and showed how much of modern mathematics could be consistently translated into a language of infinities and infinitesimals. In particular Robinson's new theory created a significant alternative to the mathematician's real-number line (alias the complete ordered field), an alternative that contained infinitesimals and on which calculus could be done in the spirit of Newton and Leibniz.

Thus by the end of the 1960's there were available on two different fronts several pretenders to what can justifiably be called the throne of mathematics: the real-number line. Although the technical construction of each of these alternatives is long and complex, there is a comparatively simple approach to both Cohen's and Robinson's models through the theory of probability; I shall now outline this approach, emphasizing its spirit more than its detail.

Before beginning this task it would be well to consider a somewhat subtle philosophical issue. If we were attempting to prove the real-number line defective and to replace it with a better model, we would have to take great care to avoid assuming the existence of the real-number line while constructing the alternative. This, however, is not what we are trying to do. We are, rather, trying to establish the existence of several different models of the real-number line, including the ordinary Platonic ideal of the complete ordered field. Therefore in constructing our new models we shall feel free to use the old one. (This process is quite analogous to the one followed in the construction of the non-Euclidean geometries, where the new models are defined within the standard Euclidean space.)

Let us now proceed to construct two specific models for the real-number system: one that will contain infinitesimal elements and another that will contain a set that violates Cantor's continuum hypothesis. Our method will be to construct a general model and then to obtain from it the two special models. The symbol $\boldsymbol{R}$ will be used throughout the discussion to stand for the ordinary real-number line; when we write $x \in \boldsymbol{R}$, we mean that x belongs to, or is a member of, the set $\boldsymbol{R}$, that is, x is a real number.

The objects in our new models will be real-valued functions f defined on some set S. In other words, f is a rule that assigns to each point s belonging to the set S a real number $f(s)$ belonging to the real-number set $\boldsymbol{R}$. The symbolism $f: S \rightarrow \boldsymbol{R}$ expresses the fact that f causes each point of S to be assigned a value in $\boldsymbol{R}$; the expression is usually read as "f maps S to $\boldsymbol{R}$" or, more formally, as "f is a function from S to $\boldsymbol{R}$." We shall use the symbol $\boldsymbol{R}^S$ to denote the set of all functions from S to $\boldsymbol{R}$.

In order for us to see how $\boldsymbol{R}^S$ can begin to resemble the real-number system, we must understand how to do addition and multiplication. Since the elements of $\boldsymbol{R}^S$ are functions (that is, rules), one must in effect define the addition and multiplication of rules. If f and g are functions in $\boldsymbol{R}^S$, we shall define $f + g$ to be the new function in $\boldsymbol{R}^S$ that assigns to each point s belonging to the set S the real number $f(s) + g(s)$; in pure symbols, $(f + g)(s) = f(s) + g(s)$. Multiplication is similar: $(fg)(s) = f(s)g(s)$. It is quite easy to show that the functions in $\boldsymbol{R}^S$ satisfy the first six axioms of a complete ordered field, listed in the illustration on the preceding page. A simple example, however, suffices to demonstrate that Axiom No. 7, the "multiplicative inverse" axiom, is not satisfied by this set [*see illustration at left*]. Thus $\boldsymbol{R}^S$ fails even to be a field and consequently falls far short of being a model for the real-number line.

To correct this situation we shall engage in a bit of mathematical gerrymandering. Axiom No. 7 does not say that all elements of $\boldsymbol{R}^S$ must have inverses; it only says that if it is true that f does not equal zero, then f must have an inverse. Accordingly if we have some functions f (that is, elements of $\boldsymbol{R}^S$) that do

1. Suppose S is the two-point set $\{0,1\}$.
2. Then each function $f{:}S \rightarrow \boldsymbol{R}$, which causes each point of S to be assigned a value in the real-number set $\boldsymbol{R}$ can be defined by giving two real numbers, namely the value of f at 0 and the value of f at 1.
3. The identity element u belonging to the trial set $\boldsymbol{R}^S$ is given by $u(0) = u(1) = 1$, and the zero element z belonging to $\boldsymbol{R}^S$ is given by $z(0) = z(1) = 0$.
4. Now the function h belonging to $\boldsymbol{R}^S$ defined by $h(0) = 0$ and $h(1) = 1$ fails to satisfy Axiom No. 7 since h is not equal to z and yet h does not have an inverse.
5. Therefore the set $\boldsymbol{R}^S$ is not an adequate model of the real-number line.

FAILURE OF $\boldsymbol{R}^S$ (the set of all functions from some arbitrary set S to the real-number set $\boldsymbol{R}$) to satisfy Axiom No. 7, the "multiplicative inverse" axiom, is demonstrated in this example. For the purpose of the demonstration S is assumed to be the two-point set $\{0,1\}$. The conclusion is that $\boldsymbol{R}^S$ falls far short of being a model for the real-number line.

not have inverses, we shall simply redefine truth so that for them the statement "f equals zero" will be true. More precisely, we shall substitute for the absolute notion of truth the more flexible concept of probable truth. In order to show how this can be carried out, I must digress for a while to discuss some definitions and examples from elementary probability.

Mathematical probability is based on a special function that assigns to each subset A of a given set Ω a positive real number that represents the probability that a point selected "at random" from the set Ω will actually be in A. This function is called a "probability measure" on the set Ω, and we shall denote it by m. The function m can be thought of as a rule that measures the size of sets, and the real number $m(A)$ can be thought of as the measure, or size, of the set A. Since the probability is 1 that a point selected at random from Ω will be in Ω, the measure of Ω must be 1. In addition we require of m only that it satisfy the (self-evident?) maxim that the whole is equal to the sum of its parts: if the whole set A is broken down into finitely or infinitely many distinct parts $A_1, A_2, \ldots, A_n, \ldots$, then $m(A) = m(A_1) + m(A_2) + \ldots + m(A_n) + \ldots$.

Before applying this measure to our set $\boldsymbol{R}^S$ I shall give two important examples of measures that will be used as the basis for our two different models for the real-number line. First, suppose Ω is the set N of positive integers: in symbols, $N = \{1, 2, 3, \ldots\}$. We shall define on N a very crude measure by classifying subsets of N as small if they are finite and large if they are cofinite, that is, if their complement (the set of positive integers not in them) is finite. Any small set will be given measure 0 and any large set will be given measure 1.

Now, the measure on N as just defined fails to measure sets such as the set of even integers, since neither it nor its complement is finite. The problem is that the set of even integers is neither small nor large but somehow in between. We can correct this deficiency in our measure m by systematically classifying each intermediate set as either small or large on an arbitrary basis, subject only to the constraints of consistency: each subset of a small set must be small; each set that contains a set previously classified as large must be large; the complement of a small set must be large, and vice versa. Once this has been done our measure will assign a value of either 0 or 1 to every subset of N; we shall call m the cofinite measure on N.

For our second example let us take for the set Ω the unit interval I that consists of all real numbers between 0 and 1; in symbols, $I = \{x \varepsilon \boldsymbol{R} \mid 0 \leqslant x \leqslant 1\}$. We build up a measure m on the subsets of I by first assigning to each interval its length. Then for a subset A that consists of pieces that are intervals we use the maxim that the whole is equal to the sum of its parts to say that the measure of A is the sum of the lengths of its pieces. Continuing in this fashion, we can build a probability measure on I that is known as the Lebesgue measure (named after the 20th-century French mathematician Henri Lebesgue).

Since each point x belonging to the set I is considered to be an interval of length 0, the Lebesgue measure m of each point x is 0. According to the construction just described, each finite set will also have measure 0, as will each infinite set that can be written as a sequence $x_1, x_2, x_3, \ldots$ (since the measure of the entire sequence is just the sum of the measures of each of its points). Since the set of rational numbers in I can be listed in a sequence (1, 1/2, 1/3, 2/3, 1/4, 2/4, 3/4, ...) it too must have measure 0, even though it appears to be a very large set. In other words, the chances are nil that a number selected at random from the unit interval will be rational.

This minor paradox leads directly to a major paradox concerning the Lebesgue measure. If I select a number at random from the unit interval, the probability is 0 that the number so chosen will equal some particular previously selected number; intuitively there are just too many numbers in the unit interval to choose from. Yet the probability that I shall pick some number in the unit interval is 1. Thus in this case it appears that the whole is not equal to the sum of its parts! This dilemma has plagued various philosophers throughout history; how can something of positive weight be made up of parts that each have zero weight?

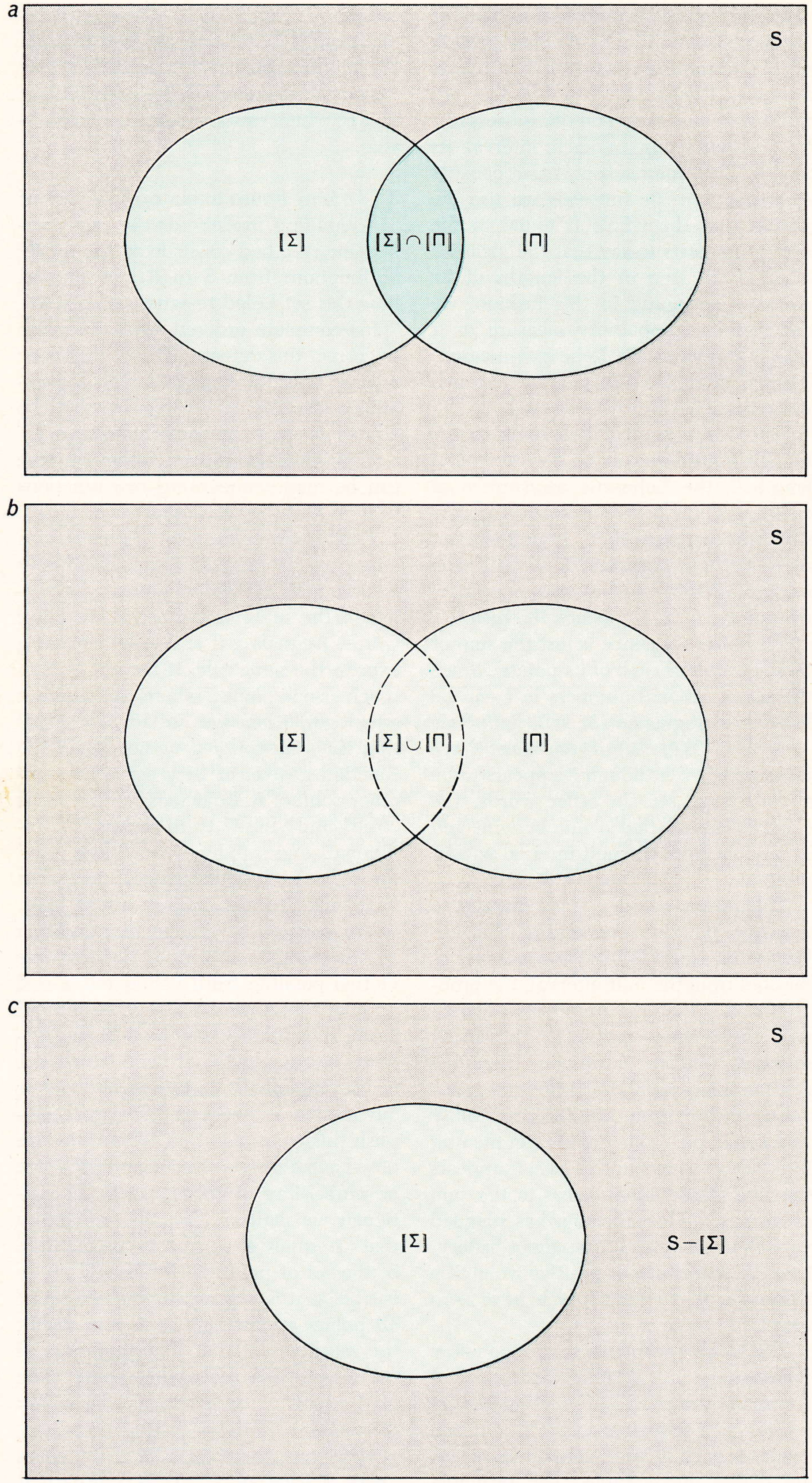
The trouble is that there are too many points in the unit interval. Cantor called countable those infinite sets that can be written as a sequence $x_1, x_2, x_3, \ldots$, one point for each integer; other infinite sets are called uncountable. The principle that the whole is equal to the sum of its parts applies only if the collection of parts is finite or countable. Hence the measure of the set of rational numbers given above is 0 because the set of rationals can be written as a sequence in which each term has measure 0. Similar reasoning does not allow us to conclude that the measure of the interval I is 0 because I cannot be written as a sequence. (The fact that the unit interval I is an uncountable set is actually a very famous theorem of Cantor's.) It follows that $\boldsymbol{R}$, which contains I, is also uncountable.

We now return to our original task of building models for the real-number line. We had constructed the set $\boldsymbol{R}^S$ of functions from S to $\boldsymbol{R}$ and showed how this set failed to satisfy Axiom No. 7 for complete ordered fields. In order to correct this defect in $\boldsymbol{R}^S$ we shall now assume that S is equipped with a probability measure m, so that we shall be able to measure the size of subsets of S.

The most primitive statement that can be made concerning two functions in $\boldsymbol{R}^S$ is that they are equal (or unequal). If f and g are given, then "f equals g" is either true or false. It is true if and only if $f(x)$ equals $g(x)$ for all points x belonging to the real-number set $\boldsymbol{R}$. In other words, f equals g if and only if f and g express the same rule. If there is in $\boldsymbol{R}$ so much as one point y where $f(y)$ does not equal $g(y)$, then we are forced to say that the function f does not equal the function g—even if they agree at every other point of $\boldsymbol{R}$. Logicians call the terms "true" and "false" truth values. In traditional logic (which we have used throughout this discussion) sentences have only two truth values: true or false.

We propose to change this tradition by assigning to sentences not just one of two possible truth values but a real number that expresses the probability of truth; this number may be 0 (if the sentence has no chance of being true) or 1 (if the sentence is certain to be true) or any number in between. We can accomplish this *coup d'état* on $\boldsymbol{R}^S$ because we now have a means of measuring the size or probability of subsets of S. In particular we shall say that the probability that "f equals g" is true is the measure of the set of points in S where it really is true, that is, the measure of the set of all points s belonging to S that satisfy the requirement that $f(s)$ equal $g(s)$. If we denote this probability truth value of "f equals g" by $|f = g|$, we have $|f = g| = m(\{s \varepsilon S \mid f(s) = g(s)\})$.

Of course, there are a lot of sentences about $\boldsymbol{R}^S$ that are far more complicated than the primitive sentences of the form "f equals g." All the more complex sentences, however, can be built up from a few primitive ones, which are called, naturally enough, atomic sentences. What we have to do to make our revolution in truth values succeed is, beginning with the atomic sentences, to systematically work our way through the



CONSIDERABLE SIMILARITY exists between the rules that govern the use in logic of the connectives "and," "or" and "not," and the operations on sets of, respectively, intersection (a), union (b) and complementation (c). This similarity (in fact, a mathematical isomorphism) was discovered by the 19th-century English mathematician George Boole, and the abstract system based on it is now called Boolean algebra. The symbol $[\![\Sigma]\!]$ stands for the set where the sentence Σ is true; $[\![\Pi]\!]$ signifies the set where the sentence Π is true. The intersection of these two sets, symbolized $[\![\Sigma]\!] \cap [\![\Pi]\!]$, is the set where the sentence "Σ and Π" is true; the union of these two sets, $[\![\Sigma]\!] \cup [\![\Pi]\!]$, is the set where "$\Sigma$ or Π" is true. In the example of complementation $S - [\![\Sigma]\!]$ signifies the set where "not Σ" is true.

entire catalogue of sentences about $\boldsymbol{R}^S$ and determine how to assign a probability to each sentence in such a way that it represents the measure of the set of points in S where the sentence is true.

The basic idea behind this scheme was outlined more than 100 years ago by the English mathematician George Boole, who noticed that there was a considerable similarity (in fact, a mathematical isomorphism) between the rules that govern the use in logic of the connectives "and," "or" and "not," and the operations on sets of, respectively, intersection, union and complementation [*see illustration at left*]. The abstract mathematical system that represents this structure is now called a Boolean algebra. To see how Boole's system can be used to solve our problem let us consider a simple example.

If Σ is a sentence about $\boldsymbol{R}^S$, we shall denote by $[\![\Sigma]\!]$ the set of all points in S where Σ is true, and by $|\Sigma|$ the measure of $[\![\Sigma]\!]$. Thus $|\Sigma| = m([\![\Sigma]\!])$ is the probability truth value of the sentence Σ. If Π is another sentence about $\boldsymbol{R}^S$, then one of the Boolean relationships is that $[\![\Sigma \text{ and } \Pi]\!] = [\![\Sigma]\!] \cap [\![\Pi]\!]$. In words, the set of points in S where both Σ and Π are true is the intersection (denoted by $\cap$) of the set where Σ is true with the set where Π is true.

Perhaps the only detail in our plan (to extend the probability truth values from atomic sentences to all sentences) that requires special comment is the treatment of the quantifiers $\forall$ (meaning "for all") and $\exists$ ("there exists"). The sentence "$\forall s \Pi(s)$," which we read as "For all points s belonging to the set S, the sentence $\Pi(s)$ is true," means that $\Pi(s_1)$ and $\Pi(s_2)$ and ... (ad infinitum) are all true. In other words, $\forall$ is an infinite repetition of "and." Thus it is appropriate that the set operation that corresponds to $\forall$ is the infinite intersection, since the ordinary (finite) intersection corresponds to "and." Similarly, $\exists$ represents an infinite "or," so that its corresponding set operation is the infinite union.

In summary, then, although we have by no means discussed all the relevant detail, it is possible by means of the Boolean translation to determine for each sentence Σ about $\boldsymbol{R}^S$ a subset $[\![\Sigma]\!]$ of S on which Σ is true. The measure of this set is the probability truth value of Σ, denoted by $|\Sigma|$. A sentence Σ will be considered valid if and only if $|\Sigma| = 1$; in other words, Σ is valid if and only if the measure of the set where Σ fails to hold is 0, so that there is "no chance" of failure. The examples of measures given above reveal the essential distinction between the concept of truth in the two-valued logic and the concept of validity in the probability-valued logic; for Σ to be true it must hold without exception for all points s belonging to the set S, but it remains valid even if it fails to hold on a fairly large subset, provided only that the measure of the exceptional set is 0.

Let us denote by $\boldsymbol{R}^S/m$ the set $\boldsymbol{R}^S$ with the new concept of validity as determined by the measure m. We have seen that under the former notion of truth $\boldsymbol{R}^S$ did not satisfy the axioms for a complete ordered field. But $\boldsymbol{R}^S/m$ will always be an ordered field because, on the basis of the concept of validity, it will always satisfy axioms No. 1 through No. 10. Furthermore, for certain choices of S and m it will also satisfy the completion axiom, so that in these cases $\boldsymbol{R}^S/m$ will be a complete ordered field.

Suppose S is the set of positive integers N with the cofinite measure m defined above in our first example of measures; m assigns to finite sets the value 0, to cofinite sets the value 1 and to intermediate sets either 0 or 1 according to some arbitrary but consistent pattern. In this case we can picture each function f belonging to $\boldsymbol{R}^S$ as a sequence $f_1, f_2, f_3, \ldots$, where for each integer i belonging to $N = S$, f_i is the real number that f assigns to i; f_i is just what we usually call $f(i)$. For example, the identity function u is the sequence $1, 1, 1, 1, \ldots$, since $u_i = u(i) = 1$ for each i belonging to $N = S$.

Given S and m as defined above, $\boldsymbol{R}^S/m$ contains both infinitesimals and infinite elements. The function f defined by the sequence $1, 1/2, 1/3, 1/4, \ldots$ is an infinitesimal, since if we multiply it by any integer n, the resulting sequence $n, n/2, n/3, n/4, \ldots$ is smaller than the identity element u except for the finite set $\{n, n/2, n/3, \ldots, n/(n-1), n/n\}$, which has measure 0. In other words, if $\Pi(n)$ is the sentence "nf is less than u," then $|\Pi(n)| = 1$ because $\{i \,\varepsilon\, N | nf(i) < u(i)\}$ has measure 1. Thus $|\forall n \Pi(n)| = 1$, which is to say that for every n, nf is less than u. Thus f is an infinitesimal since only an infinitesimal could have this property.

For a different perspective let us look at the multiplicative inverse of f; f must have an inverse since $\boldsymbol{R}^S/m$ is a field. In reality the function g defined by the sequence $1, 2, 3, 4, \ldots$ is the inverse of f since, clearly, $gf = u$. Just as f is an infinitesimal, so g must be infinite. In fact, if n denotes the constant function $n, n, n, \ldots$, then $|g > n| = m(\{i | g_i > n_i\}) = m(\{i | i > n\}) = 1$, since the set in question is cofinite. Thus in $\boldsymbol{R}^S/m$ it is valid to say that g is greater than n for all n. This is just what we mean when we say that g is infinite.

We have now achieved the first of our two goals: the description of a mathematically consistent model of the real numbers that contains infinitesimals. The other type of model, in which Cantor's continuum hypothesis fails, requires some more work. First, however, I shall explain Cantor's hypothesis.

Cantor developed his theory of cardinal numbers by defining two sets as being of equal size (or of the same "cardinality") if there is some function, or rule, that establishes a one-to-one correspondence between them. All finite sets with the same number of elements have the same cardinality, and all infinite sequences have the same cardinality as the set of integers. (The sets in this latter category are those that we have called countable.) Cantor's great achievement was to assign sizes (cardinal numbers) even to the uncountable sets. In particular he identified a class of sets of the same cardinality as the set of real numbers $\boldsymbol{R}$; he called the cardinal number of sets in this class c, for continuum. Sets of size c are those that can be put into one-to-one correspondence with the real numbers.

Cantor's continuum hypothesis is simply this: Every infinite set of real numbers is either countable or of cardinality c. There are no sets of intermediate size. This hypothesis was formulated (but not proved) by Cantor late in the 19th century, and it was listed in 1900 by David Hilbert as the first of his celebrated 23 problems for 20th-century mathematics. We shall now outline a version of Cohen's 1963 model that shows conclusively that Cantor's continuum hypothesis could never be proved from the ordinary axioms of set theory and real numbers.

This new model, developed principally by Dana S. Scott of Princeton University, involves the use of a rather complicated set S in the general model $\boldsymbol{R}^S/m$. We first have to find a very large set T, a set whose cardinality is bigger than c, the cardinal number of $\boldsymbol{R}$. The proof that such sets exist is another of Cantor's significant contributions to mathematics. An example of such a set is what is called the power set of $\boldsymbol{R}$: the set that consists of all subsets of $\boldsymbol{R}$. Cantor's proof that the cardinal number of this set is indeed bigger than c is essentially the same as his proof that $\boldsymbol{R}$, or equivalently the unit interval I, is strictly bigger than the set of integers.

Let I denote the unit interval ($I = \{x \,\varepsilon\, \boldsymbol{R} \,|\, 0 \leqslant x \leqslant 1\}$), and let T be a set whose cardinal number is bigger than c. Then we define S to be the set $I^T = \{f : T \to I\}$. The points of S are functions from T to I. The measure on S, whose details need not concern us, is a rather natural extension to I^T of the Lebesgue measure on I described above. Our new model for the real-number line is $\boldsymbol{R}^S/m$, where S equals I^T.

The fact that the points of S are really functions can be the source of some conceptual difficulty if we confuse the functions in $S = I^T$ with those in $\boldsymbol{R}^S$. Mathematicians avoid this confusion by emphasizing in their minds that S is a *set* of functions rather than a set of *functions*. We put up with this double meaning because the entire key to Scott's model is a deliberate pun formed by interchanging the role of functions and points.

To be precise, we observe that each point t belonging to the set T can be thought of as a function in $\boldsymbol{R}^S$ that assigns to each function f belonging to $I^T = S$ the real number $f(t)$ belonging to I. If we call the function t by the name $\hat{t}$, then $\hat{t}(f)$ equals $f(t)$ whenever f is a member of S. Now our problem is to find a subset of $\boldsymbol{R}^S$ that is uncountable, yet not as large as $\boldsymbol{R}^S$. But just how large is $\boldsymbol{R}^S$? Since it contains a $\hat{t}$ for each t belonging to T, the cardinality of $\boldsymbol{R}^S$ must be at least as large as that of T, but T was selected so that its cardinality was larger than c. Hence the cardinality of $\boldsymbol{R}^S$, our new real-number line, is greater than c, the cardinality of the old real-number line $\boldsymbol{R}$.

Since the new real-number line is so very much bigger than the old one, it will be much easier to find in the new line an uncountable set that is not of the same cardinality as the entire line. All we have to do is select a subset $\hat{P}$ of T with the property that the cardinality of P is c. (If we had been thinking of T as the power set of $\boldsymbol{R}$, then we could take $\boldsymbol{R}$ itself for P.) Then the set P that consists of all the functions $\hat{t}$ for t belonging to P is a subset of $\boldsymbol{R}^S$ of size c. Thus $\hat{P}$ is uncountable, yet it is not as large as $\boldsymbol{R}^S$.

This completes our program of constructing new models of the real-number line. To summarize briefly, each model consists of functions from some set S to the ordinary real-number line $\boldsymbol{R}$. The two-valued true-false logic of the old system is replaced with a more flexible probability-valued logic determined by some measure on the set S. Validity in the new model means truth with probability 1, a definition that grants exceptions to sets of measure 0. The particular models that contain infinitesimals or counterexamples to the continuum hypothesis are then formed by selecting special sets S with special probability measures m.

In the preceding development of the two new models of the real-number line I not only omitted all proofs and technical detail but I also failed to mention a most important distinction between these two models: the Cohen-Scott model (where S equals I^T) is a complete ordered field, whereas the Robinson model (where S equals N) is just an ordered field. The fact that Robinson's model fails to be complete should come as no surprise, for we did see that any real-number line that contained infinitesimals could not satisfy Axiom No. 11. To complete the discussion of these models I should explain two things: Why should the two models behave differently with respect to Axiom No. 11, and by what right can we call an object (such as Robinson's model) that fails to satisfy Axiom No. 11 a real-number line?

The answers to both questions depend on a distinction that logicians make between what are called first-order sentences and higher-order sentences. Intuitively a first-order sentence about the real numbers is one that makes generalizations only about real numbers and not about such higher abstractions as functions or sets of real numbers; in technical terms a first-order sentence is one in which the quantifiers ($\forall$ and $\exists$) are limited to real numbers. All the axioms of a complete ordered field are first-order sentences except for the completion axiom (No. 11), which talks about a property of *all* subsets.

Now, the models of nonstandard analysis (such as the first of our two examples) have the property that every first-order sentence that is true about $\boldsymbol{R}$ is true about them, and conversely. In other words, it is impossible to distinguish between these new models and $\boldsymbol{R}$ by the use of first-order sentences. Since a large part of calculus can be expressed in first-order sentences, it is quite literally true that for most practical purposes the nonstandard real-number line is the same as the ordinary real-number line.

The effort required to show that the completeness axiom is satisfied by some particular model $\boldsymbol{R}^S/m$ is decidedly more strenuous than any of the ideas discussed above. Not only is the axiom itself more difficult to verify, but also before we can even consider it we must work quite hard to extend the machinery of assigning probability truth values from first-order sentences to higher-order sentences. This extra effort is also required to prove that in our second example the probability is zero that the continuum hypothesis is true—since the continuum hypothesis is also a higher-order sentence.

There is a striking analogy between the creation of these new models for the real-number line and the development of quantum mechanics. The most common derivation of quantum mechanics from classical mechanics is accomplished by substituting in the classical theory a wave packet or probability cloud for each classical particle. In the quantum model the location of the particle in the cloud is given by a probability distribution and the particle itself becomes a random variable.

The new models for the real-number line that were described above are formed in just this way. We substituted for the ordinary real numbers certain real-valued functions defined on a probability space. These functions are precisely what statisticians and probability theorists call random variables. The substitution of random variables for real numbers was accompanied by a corresponding change from a two-valued logic to a more general scheme in which the truth value of a statement about the real numbers is expressed by a probability. This transition from two-valued to probability-valued logic also occurs in the derivation of quantum mechanics from classical mechanics, since questions about the state of the mechanical system that in the classical framework could be answered by "Yes" or "No" can in quantum theory only be answered with a probability.

One of the best-known consequences of quantum mechanics is Werner Heisenberg's uncertainty principle, which places an absolute limit on the amount of information that can be obtained by a physicist through the process of observing a particle. Heisenberg's principle is derived by a process of self-reflection in which the observer analyses the effect that his actions have on the system he is observing.

In mathematics, analogously, one deduces from the action of writing down axioms and theorems (which is to the mathematician what experimental observation is to the physicist) a limitation on the amount of information that can be derived from a set of axioms (in the Gödel-Cohen case) and a limitation on the capability of axiom systems to describe models uniquely (in Robinson's theory). The first of these two limitations is the substance of Gödel's undecidability theorem, whereas the second is one version of a more general result known in mathematical logic as the Lowenheim-Skolem theorem. In different ways each limitation expresses a basic uncertainty in mathematics.

It is not surprising that the introduction of "uncertainty" results in mathematics was accompanied by fundamental

$\boldsymbol{R}$ =	Real-number line
ϵ =	"Belongs to"; "is a member of" (as in $x \epsilon \boldsymbol{R}$, that is, x belongs to, or is a member of, the set of real numbers $\boldsymbol{R}$; hence x is a real number)
$f{:}S \rightarrow \boldsymbol{R}$ =	A function f that causes each point of the set S to be assigned a value in the real-number set $\boldsymbol{R}$
$\boldsymbol{R}^S$ =	The set of all functions from S to $\boldsymbol{R}$
$m(A)$ =	The probability measure of the set A
Σ =	A sentence about $\boldsymbol{R}^S$
$[\![\Sigma]\!]$ =	The set of all points in S where Σ is true
$\lvert\Sigma\rvert$ =	The truth value of the sentence Σ, that is, the probability measure of $[\![\Sigma]\!]$; also written $m([\![\Sigma]\!])$
Π =	Another sentence about $\boldsymbol{R}^S$
$\cap$ =	Intersection of two sets
$\cup$ =	Union of two sets
$\forall$ =	"For all"
$\exists$ =	"There exists"
c =	Cantor's symbol for the cardinal number of the continuum

GLOSSARY OF SYMBOLS used in this article is provided in this illustration. The symbol c, used by Georg Cantor to signify continuum, denotes the cardinal number of sets in the class of infinite, uncountable sets of the same cardinality, or size, as the set of real numbers $\boldsymbol{R}$.

disagreement about their significance, just as Heisenberg's uncertainty principle precipitated among physicists a major dispute about its ultimate significance. Most physicists tend to agree with Niels Bohr that the uncertainty in quantum mechanics is a fundamental law of nature. Some, however, support the position of Albert Einstein, who argued that the uncertainty principle merely expresses a limitation on the present conceptual formulation of physics.

These two opposite poles in physics reflect approximately the positions of the Formalist and the Platonist in mathematics. The Formalist believes that mathematics is pure form, and since the "uncertainty" theorems of mathematics limit the scope and power of the formalism with which mathematics is concerned, then they must forever limit mathematics itself. For a Formalist the question of whether the "real" real-number line satisfies the continuum hypothesis or contains infinitesimals is as meaningless as is, for a physicist such as Bohr, the question of whether or not an electron "really" has an exact simultaneous position and velocity. Robinson himself reflects this position when he writes (in a paper titled "Formalism 64") that "any mention of infinite totalities is literally meaningless."

In contrast, the Platonists, who count among their number even Gödel himself, believe (like Einstein) that the undecidability in mathematics is a statement about the inherent limitations of our present axiomatic mode of investigation and not about the mathematical objects themselves. Gödel has argued that there is no reason to believe more in the objective existence of a physical object such as an electron than in the objective existence of a mathematical object such as the real-number line. And one who believes that the real-number line has an existence in a Platonic universe can hardly avoid wondering about which of our several models is the more accurate description of this Platonic continuum.

It is ironic that Robinson, the re-creator of infinitesimals, does not believe they really exist, whereas Gödel, the prophet of undecidability, believes in a Platonic universe in which the properties of mathematical objects are visible for those who have the eyes to see. Perhaps these men are merely reflecting an intense modesty about the significance of their own achievements. It seems unlikely, however, that within the next few generations mathematicians will be able to agree on whether every mathematical statement that is true is also knowable.