WHAT IS A SHEAF?

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Ever since Jean Leray and Henri Cartan in 1950 formally introduced the concept of a sheaf, the various examples and applications of sheaves have come to play a major role in such diverse fields as several complex variables, algebraic geometry, and differential and algebraic topology. Yet nearly all monographs which use or introduce sheaves assume the sophistication of graduate level algebraic topology. So it is very difficult for an undergraduate to acquire from the available literature a real understanding of sheaves and their applications. It is the purpose of this article to introduce the theory of sheaves at an elementary level with the hope that the interested reader will then be able to approach any of the standard treatises (e.g., [2], [3], [6], [10], or [11]) with significant insight.

Our avenue of approach to the theory of sheaves will be through examples drawn from three major areas of mathematics: from analysis, the sheaf of germs of holomorphic functions; from algebra, the sheaf of local rings; and from geometry, the sheaf of differential forms. We will develop each of these particular sheaves in considerable detail, for the different perspectives thus revealed will more readily make transparent the subsequent discussion of the general theory of sheaves.

1. The sheaf of germs of holomorphic functions. A *holomorphic* (or *analytic*) function on an open subset D of complex n-space C^n is defined to be a complex valued function on D which has a local power series representation at each point of D. Osgood's lemma [6, p. 2] asserts that a continuous function on $D \subset C^n$ is holomorphic if and only if it is holomorphic in each variable separately.

A very important property of holomorphic functions is that they are uniquely determined by their behavior on open sets: if f and g are holomorphic on a domain D (a *domain* is a connected open set), and if f equals g on a nonempty open subset of D, then f equals g on all of D. To see this, we need only observe that the largest open subset of D on which $f=g$ is also closed (relative to D), since the partial derivatives which determine the power series expansion are continuous. Since D is connected, this set must be D.

Now if $z \in C^n$, we say that f is *holomorphic at* z if it is holomorphic on some neighborhood of z. The collection A_z of functions holomorphic at z forms an algebra over the field of complex numbers in which the operations of sum and

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product involve intersection of domain: if $f: U\to C$ and $g: V\to C$, then $f+g: U\cap V\to C$ and $fg: U\cap V\to C$. We let I_z be the ideal in A_z consisting of those functions in A_z which vanish identically on some neighborhood of z.

The *algebra of germs of holomorphic functions* at z is then defined to be the quotient ring (algebra) A_z/I_z, and is denoted by O_z. So a germ of a holomorphic function is an element $f+I_z$ of O_z, where f is holomorphic at z. We will usually denote this germ by $[f]_z$. Following the usual practice, we shall often identify, or fail to distinguish between, two functions which belong to the same germ. This sloppiness is somewhat justified by the uniqueness property stated above, for two functions which belong to the same germ and are defined on the same domain D must differ by a function in I_z, which means that they agree on some neighborhood of z, and thus must agree on D.

We may now define the *stalk space* (*espace êtalê*) of germs of holomorphic functions to be the set $S=\{(z,[f]_z)\mid f \text{ is holomorphic at } z\in C^n\}$ together with the natural mapping ρ from S to C^n defined by $\rho((z, [f]_z))=z$. We call $\rho^{-1}(z)$ the *stalk* at the point $z\in C^n$; it is simply a copy of O_z, the algebra of germs of holomorphic functions at z. The stalk space S is thus the disjoint union of the stalks. Intuitively, we shall picture S as a space of interpenetrating sheets lying over C^n, with ρ projecting S onto C^n (Fig. 1).

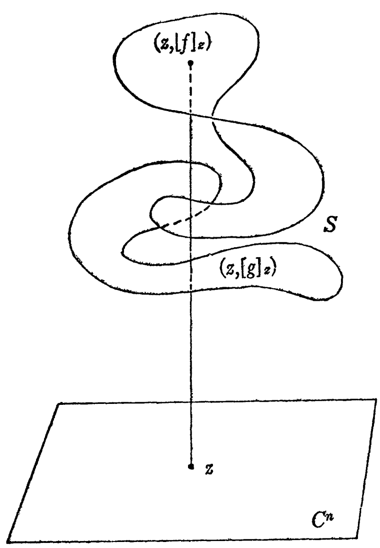

FIG. 1

To make this intuitive picture more precise, we lift the topology of C^n back to S, to make S into a topological space. For each open set U in C^n and each function f which is holomorphic on U, we define $V(f, U)=\{(z,[f]_z)\mid z\in U\}$. Each such $V(f, U)$ is contained in S, and the collection of all such sets covers S, for if $(z, [f_0]_z)\in S$, f_0 must be holomorphic on some neighborhood U_0 of z and $(z, [f_0]_z)\in V(f_0, U_0)$. Furthermore, $V(f_1, U_1)\cap V(f_2, U_2)=V(f, U)$ where $U=\{z\in U_1$

$\cap U_2 | [f_1]_z = [f_2]_z\}$ and $f = f_1|_U = f_2|_U$. Thus the sets $V(f, U)$ form a basis for a topology on S, and relative to this topology, the projection ρ is a *local homeomorphism*. That is, for each basis neighborhood $V(f, U)$ in S, the one-to-one map $\rho|_{V(f,U)}$ is a homeomorphism onto U. For if we let $\rho_{f,U}$ denote $\rho|_{V(f,U)}$, and if N is an open subset of U, then $\rho_{f,U}^{-1}(N) = V(f, N)$ which is open in S, while if $V(f, U') \subset V(f, U)$, then $\rho_{f,U}(V(f, U')) = U'$. The topology on S is uniquely determined by the requirement that the projection ρ be a local homeomorphism.

This topological space S, together with the local homeomorphism ρ which projects S onto C^n, is called the *sheaf of germs of holomorphic functions* over the *base space* C^n. As the agricultural terminology implies, we think of the sheaf as a bundle of stalks (Fig. 2), each with a full head of germs (or, if you wish, seeds, or grain).

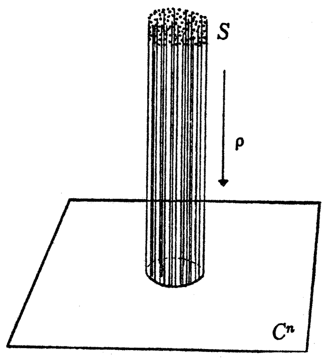

FIG. 2

We can show that the stalk space S is Hausdorff as follows: points of S may differ either because they are on different stalks, or because they are on different levels of the same stalk. In the first case, the projections of two points $p, q \in S$ differ in C^n; so, since C^n is Hausdorff, there are disjoint neighborhoods of $\rho(p)$ and $\rho(q)$ which may be lifted back to S. To be specific, if $p = (z, [f]_z)$ and $q = (w, [g]_w)$ where $z \neq w$, then there exist disjoint open neighborhoods U_z, U_w of z and w respectively and on them holomorphic functions $f \in [f]_z$ and $g \in [g]_w$, respectively, so that $V(f, U_z)$ and $V(g, U_w)$ are disjoint neighborhoods of p and q.

The second case is a bit more complex, since it depends on the uniqueness property of holomorphic functions. If $p = (z, [f]_z)$ and $q = (z, [g]_z)$ are different points on the same stalk, then $[f]_z \neq [g]_z$; so there must exist different holomorphic functions $f \in [f]_z$ and $g \in [g]_z$ which are both defined on some neighborhood U of z. We claim that $V(f, U)$ and $V(g, U)$ are then disjoint neighborhoods of p and q, for if $(w, [h]_w) \in V(f, U) \cap V(g, U)$, then $w \in U$ and $[f]_w = [h]_w = [g]_w$. But as we observed above, the uniqueness property of holomorphic functions implies that the two functions f and g with the same domain U which belong to the same germ $[h]_w$ must be identical on U. But they are not identical on U, since $[f]_z \neq [g]_z$. So $V(f, U) \cap V(g, U) = \varnothing$, and thus S is Hausdorff.

There is still another consequence of the uniqueness property of holomorphic functions that can be used to further illuminate the sheaf of germs of holomorphic functions. The uniqueness property may be roughly interpreted as saying that the global behavior of a holomorphic function is uniquely determined by its behavior on any open set. This makes meaningful the vague question of identifying the largest domain to which a given holomorphic function can be extended. In the classical study of analytic functions this question led to the concept of a Riemann surface, or more generally to complex analytic manifolds.

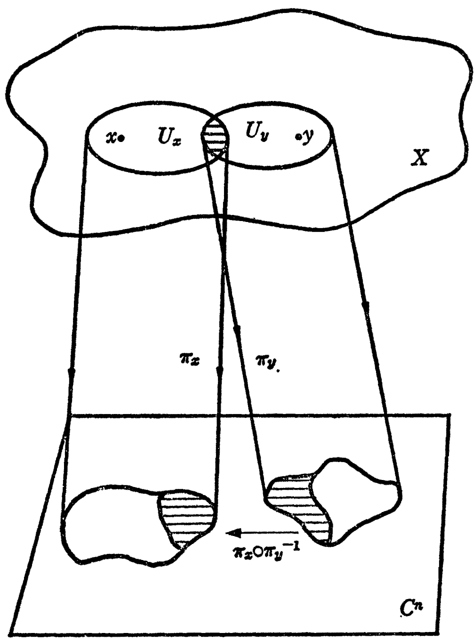

FIG. 3

A manifold is, essentially, a topological space which is locally homeomorphic to complex Euclidean n-space C^n. To be more precise, we will call a topological space X *locally Euclidean* (of dimension n) if every $x \in X$ is contained in an open set U_x which is homeomorphic under a mapping π_x to some subset of C^n, where, furthermore, the *coordinate patches* U_x are coherent in the sense that for each $x, y \in X$, $\pi_x \circ \pi_y^{-1}$ is a homeomorphism between $\pi_y(U_x \cap U_y)$ and $\pi_x(U_x \cap U_y)$. The first half of this definition guarantees that X is locally like C^n, while the second condition requires that the locally Euclidean patches overlap so as to form a coherent Euclidean structure on all of X. We shall call each pair (U_x, π_x) a *local coordinate system*, since π_x^{-1} lifts the coordinate system of C^n back to U (Fig. 3).

Since the same topological space may be covered by several different collections of coordinate systems (U_x, π_x), and since we do not wish to distinguish between two covers which provide essentially the same coordinate structure on X, we define a *manifold* to be a locally Euclidean topological space in which the collection of coordinate systems (U_x, π_x) is maximal with respect to the defining properties for a locally Euclidean space. Since each locally Euclidean space generates a unique manifold, we shall often refer to locally Euclidean spaces as manifolds even if the collection of local coordinate systems is not maximal. Other types of manifolds may be produced by projecting to real Euclidean space R^n instead of to C^n or by requiring that the homeomorphisms $\pi_x \circ \pi_y^{-1}$ be analytic or C^∞ (infinitely differentiable); such manifolds are naturally called *analytic manifolds* or *C^∞ manifolds.*

A function $f: X \to Y$ from one analytic manifold to another is called *holomorphic* if for each x and y, $\pi_y \circ f \circ \pi_x^{-1}$ is holomorphic on its domain, which is $\pi_x(U_x \cap f^{-1}(U_y))$. In the special case, where $Y = C^1$, the identity map $i: Y \to C^1$ is used to define the local coordinate systems. So a holomorphic function f from the analytic manifold X to C^1 is characterized by the property that $f \circ \pi_x^{-1}$ is holomorphic on $\pi_x(U_x)$.

It should be clear from this description that the sheaf of germs of holomorphic functions can be regarded as an analytic manifold, using the projection ρ to define the local coordinate systems. It is a particularly important manifold, since on it we can define what is known as the *universal holomorphic function.* This is the mapping $F: S \to C$ defined by $F((z, [f]_z)) = f(z)$. F is clearly holomorphic since for each local coordinate system $(U, \rho|_U)$, we have $F \circ (\rho|_U)^{-1} = f|_{\rho(U)}$ where $f: \rho(U) \to C$ is holomorphic.

$$
\begin{tikzcd}
& S \arrow[dr, "F"] & \\
& D \arrow[u, "\hat{f}"] \arrow[r, "f"'] & C
\end{tikzcd}
$$

FIG. 4

F is universal in the sense that the behavior of all holomorphic functions on C^n is subsumed in that of F. In particular, whenever $f: D \to C$ is holomorphic (where D is a domain in C^n), we can factor f through the sheaf S as follows: there exists a unique function $\hat{f}: D \to S$ such that $F \circ \hat{f} = f$. Clearly $\hat{f}$ is defined by $\hat{f}(z) = (z, [f]_z)$, so the associated diagram (Fig. 4) is commutative; $\hat{f}$ is continuous since $\hat{f}^{-1}(V(f, U)) = U$.

With this structure, we can now describe the *domain of holomorphy* of a given holomorphic function f—that is, the largest domain to which f can be uniquely extended. It is the connected component E of S which contains $\hat{f}(D)$. (Of course $\hat{f}(D)$ is connected since it is the continuous image of a connected set.) Although D is not literally a subset of E, it is imbedded by $\hat{f}$ in E, and thus the universal holomorphic function F is the extension to E of the function f.

2. The sheaf of local rings. Let A be a commutative ring with 1, and S a multiplicatively closed nonempty subset of A with $0 \notin S$. We construct from A and S a ring A_S called a *ring of quotients of* A, in which the elements of S have multiplicative inverses. On the set

$$A \times S = \{(a, s) \mid a \in A, s \in S\}$$

we define an equivalence relation $(a, s) \sim (b, t)$ if and only if there exists $r \in S$ such that $(at - bs)r = 0$. We also define two operations,

$$(a, s) + (b, t) = (at + bs, st) \quad \text{and} \quad (a, s)(b, t) = (ab, st),$$

which are compatible with the relation. We denote by A_S the ring of equivalence classes with the induced operations. As in the ring of integers with the set of nonzero elements as S, the equivalence class of (a, s) is denoted by a/s; thus we call S the *set of denominators*.

The 0 of A_S is $0/s$ (any s in S), the identity is s/s; and if $s \in S$, $s^{-1} = 1/s$. There is a homomorphism $\alpha: A \to A_S$ defined by $\alpha(a) = as/s$, which is independent of the choice of s. Of course if A is an integral domain, α is one-to-one because $\operatorname{Ker} \alpha = \{a \mid sa = 0 \text{ for some } s \in S\}$.

If I is an ideal of A, the ideal $\alpha(I)$ in A_S can be represented by

$$\alpha(I) = \{a/s \mid a \in I, s \in S\},$$

and we shall write IA_S for $\alpha(I)$. This function α on the set of ideals of A defines a one-to-one correspondence between the set of prime ideals in A_S and the set of prime ideals in A whose intersection with S is empty [**12**, p. 223].

Since we may describe a prime ideal P in A as one whose complement is multiplicatively closed, we may form the ring of quotients of A whose set of denominators is the complement of P. We shall denote this ring of quotients by A_P, and call it the *local ring* of A at P. The ring A_P has only one maximal ideal, PA_P, since clearly P is the largest prime ideal of A with the property that its intersection with the complement of P is empty.

These local rings will be the stalks for the sheaf of local rings and the set of prime ideals of A will form the base space. This space is called the *spectrum* of A, denoted by Spec A, and is topologized by taking as a basis for the topology all sets $V_x = \{P \in \operatorname{Spec} A \mid x \notin P\}$ where $x \in A$. Then $V_1 = \operatorname{Spec} A$, $V_0 = \varnothing$, and $V_x \cap V_y = V_{xy}$; thus $\{V_x\}_{x \in A}$ is a basis. Since $\bigcup_{x \in M} V_x = \{P \mid (x)_{x \in M} \not\subset P\}$ where $(x)_{x \in M}$ is the ideal generated by the subset M of A, any ideal I of A defines an open set $V_I = \{P \mid I \not\subset P\}$, and every open set U is of this form although I is not uniquely determined by U. A closed set, then, is a set of primes containing some fixed ideal, so that a point P in Spec A is closed if and only if it is a maximal ideal. For most rings, therefore, the base space is not even T_1; however, Spec A is always T_0.

It is possible to define a ring of quotients with respect to the complement of a prime P because it is multiplicatively closed, but the complement of a union of primes is also multiplicatively closed. Hence, with each nonempty open set U in Spec A we may associate the ring of quotients $A_U = \{a/s \mid P \in U \Rightarrow s \notin P\}$ whose set of denominators is the complement of the union of all the primes in U. If U and V are open sets in Spec A such that $U \subset V$, we may define *restriction homomorphisms* $\rho_{U,V}: A_V \to A_U$ as follows: if $a/s \in A_V$, s is not an element of any prime ideal P in V, and so *a fortiori* not an element of any prime ideal P in U. Hence a/s is also an element of A_U. We define $\rho_{U,V}(a/s) = a/s$, but this map is not the identity, or even one-to-one, since the equivalence classes which are used to define the ring A_U are larger than those used to define A_V. The kernel of $\rho_{U,V}$ consists of those elements a/s such that a is a zero divisor with respect to an element in one of the prime ideals of V which is not in any element of U.

An important property of $\rho_{U,V}$ is the commutativity of the diagram in Fig. 5, where $\alpha_U: A \to A_U$ takes a to $a/1$. Now $\rho_{U,V}$ is uniquely determined by this property and from this it follows that $\rho_{U,U}$ is the identity map and that if $U \subset V \subset W$, then $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$. This system, consisting of Spec A, the rings A_U, and maps $\rho_{U,V}: A_V \to A_U$ when $U \subset V$, is called a *presheaf* over Spec A. Besides the maps $\rho_{U,V}$ corresponding to pairs of open sets for which $U \subset V$, we

$$\begin{tikzcd} & A_V \arrow[dd, "\rho_{U,V}"] \\ A \arrow[ur, "\alpha_V"] \arrow[dr, "\alpha_U"'] & \\ & A_U \end{tikzcd}$$

FIG. 5

$$\begin{tikzcd} & A_V \arrow[dd, "\rho_{P,V}"] \\ A \arrow[ur, "\alpha_V"] \arrow[dr, "\alpha_P"'] & \\ & A_P \end{tikzcd}$$

FIG. 6

can define maps $\rho_{P,V}: A_V \to A_P$ when $P \in V$. If V is open and $P \in V$, we define $\rho_{P,V}: A_V \to A_P$, by $\rho_{P,V}(a/s) = a/s$, which is possible since $a/s \in A_V$ implies $s \notin P$. The map $\rho_{P,V}$ is the unique map which makes the associated diagram commute (Fig. 6). As before, it follows from the uniqueness of $\rho_{P,V}$ that

$$\rho_{P,W} \circ \rho_{W,U} = \rho_{P,U} \qquad \text{if } P \in W \subset U.$$

If U is open in Spec A, and u is any element of A_U, we may treat u as a function from U to the stalk space $S = \bigcup \{A_P \mid P \in \text{Spec } A\}$ by defining $u(P) = \rho_{P,U}(u) \in A_P$ for $P \in U$. If $V \subset U$, we call $\rho_{V,U}(u)$ the *restriction* of u to V. If $u \in A_U$ and $v \in A_V$, and if there is an open set $W \subset U \cap V$ such that $\rho_{W,U}(u) = \rho_{W,V}(u)$, we say u and v *agree on the open set* W, for if $P \in W$, then

$$u(P) = \rho_{P,U}(u) = \rho_{P,W} \circ \rho_{W,U}(u) = \rho_{P,W} \circ \rho_{W,V}(v) = \rho_{P,V}(v) = v(P).$$

If $u \in A_U$, either the function u or its image $u(U)$ in S is called a *section* of the presheaf. The set of sections $u(U)$ covers S, for if $a/s \in A_P$ then s is a denominator in A_V, where $V = \{Q \mid s \notin Q\}$, and $a/s \in A_V$ is a section over V whose image at P is a/s.

The collection of all these sections $u(U)$ is a basis for a topology on S, since the intersection of two sections is a union of sections. For, suppose $u \in A_U$ and $v \in A_V$ are sections over the open sets U and V. If $u(U) \cap v(V) = \varnothing$ there is nothing to show. If $x \in u(U) \cap v(V)$, $x = \rho_{P,U}(v) = \rho_{P,V}(v) \in A_P$ for some $P \in U \cap V$. Then $u = a/s$, where s is not an element of any element of U, and $v = a'/s'$, where s' is not an element of any element of V. Since $a/s = a'/s'$ in A_P, there exists $t \notin P$ such that $t(as' - a's) = 0$. Let $W = \{Q \mid t \notin Q\}$. Then $a/s = a'/s'$ in $A_{W \cap U \cap V}$ and, since the diagram in Fig. 7 commutes, the section over $W \cap U \cap V$ defined by $a/s = a'/s'$ is a subset of $u(U)$ and of $v(V)$ and it is a neighborhood of x.

$$\begin{tikzcd} A_V \arrow[dr] \arrow[drr] & & \\ & A_{W \cap U \cap V} \arrow[r] & A_P \\ A_U \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

FIG. 7

In this topology the projection ρ is a local homeomorphism, for if $x \in A_P$ there is a section v over V through x for some V, and the restriction of ρ to the section $v(V)$ is one-to-one and onto. If $U \subset V$ is open and $x \in U$, the restriction of v to U is also a section through x, and thus $u(U)$, where $u = \rho_{U,V}(v)$, is open in $u(V)$; so ρ is continuous. If N is an open subset of $v(V)$ it is a union of sections over open subsets of V and $\rho(N)$ is the union of these open subsets.

The topological space S, together with the projection $\rho: S \to \text{Spec } A$, is called the *sheaf of local rings* over Spec A. If U is an open subset of Spec A, any continuous function $f: U \to S$ such that $\rho \circ f$ is the identity on U is called a *section* of the sheaf S; the set of all sections over the open set U is denoted by $\Gamma(U, S)$. The relation between the sections of the sheaf S and the sections of the presheaf (that is, the elements $u \in A_U$) is rather subtle, for even though each presheaf section may be thought of as a (continuous) function on U which is a local inverse for ρ, two anomalies may occur. It may be that two different elements $u = a/s$ and $v = b/t$ of the ring A_U yield the same function under the interpretation outlined above, or there may be functions in $\Gamma(U, S)$ which cannot be derived from any section $u \in A_U$. Thus the interpretation map from A_U to $\Gamma(U, S)$ need not be either one-to-one or onto, though if U is a basis set (that is, one of the form V_x for some $x \in A$) this map is an isomorphism [**5**, No. 4, p. 86]. For this reason, the open sets of the form V_x are often called *distinguished* open sets.

If the base space Spec A is not Hausdorff, certainly the sheaf S cannot be Hausdorff. But even if Spec A is T_2 it may well happen (although it is difficult to visualize it on Hausdorff paper) that two distinct sections over U, u and u', may agree on a proper open subset V of U (Fig. 8). If P is in the closure of V but not in V, $u(P)$ and $u'(P)$ are distinct points of the sheaf but cannot be separated because every neighborhood of P contains points of V on which u and u' agree.

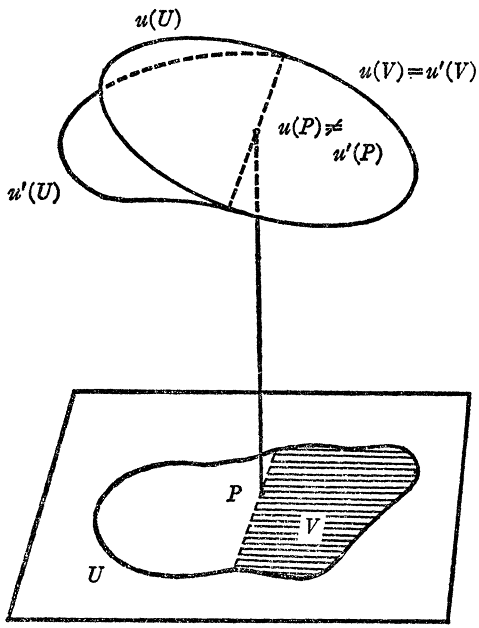

FIG. 8

A special case. In order to provide a more geometric interpretation of the sheaf of local rings as well as a glimpse of the origin of the subject, we shall look at some special rings. Let Q be the field of rational numbers, $Q[x_1, \cdots, x_n]$ the ring of polynomials in n variables with rational coefficients, and I an ideal of $Q[x_1, \cdots, x_n]$. Let A be the quotient ring $Q[x_1, \cdots, x_n]/I$, which we may write $Q[\bar{x}_1, \cdots, \bar{x}_n]$, where $\bar{x}_i = x_i + I$, $i = 1, \cdots, n$. If $E = \{(x_1, \cdots, x_n) \in C^n \mid p(x_1, \cdots, x_n) = 0$ for all $p \in I\}$, i.e., E is the intersection of the zero sets of all polynomials in the ideal I, we may interpret A as a ring of complex valued functions on E. For since $p_1(x) - p_2(x) \in I$, whenever $p_1(\bar{x}) = p_2(\bar{x})$, $p_1(\bar{x}) = p_2(\bar{x})$ implies $p_1(x) = p_2(x)$ if $x = (x_1, \cdots, x_n) \in E$.

We may also construct, from the pairing which takes $p \in A$ and $x \in E$ to $p(x) \in C$, a one-to-one correspondence between the points in E and homomorphisms from A to C. If $x = (x_1, \cdots, x_n) \in E$ the function which takes p to $p(x)$ is a homomorphism from A to C, and conversely, if $\phi: A \to C$ is a homomorphism, $x = (\phi(\bar{x}_1), \cdots, \phi(\bar{x}_n)) \in E$, since if $p \in I$,

$$p(x) = p(\phi(\bar{x}_1), \cdots, \phi(\bar{x}_n)) = \phi(p(\bar{x}_1), \cdots, p(\bar{x}_n)) = \phi(0) = 0.$$

Since C is an integral domain, the kernel of the homomorphism defined by $x \in E$ is a prime ideal in A. Conversely, it may be shown that if P is an element of Spec A it is the kernel of some homomorphism from A to C [13, p. 164]. So we may regard Spec A as a set of equivalence classes of E under the relation $x \sim x'$ if and only if x and x' are zeros of the same polynomials $p \in A$. Thus if $x \sim x'$, $p(x) = p(x')$ for all p in A, so A may also be interpreted as a ring of functions on Spec A.

That an element p of A not be in a prime P means that p is not in the kernel of the map $A \to A/P$, or that p is not zero on any point x in E for which the kernel of the corresponding homomorphism is contained in P. Thus, the local ring A_P at P, which contains the inverses for all $p \notin P$, is the ring of all rational functions defined locally, i.e., in some neighborhood of P in Spec A. The set of such functions which vanish at P is the only maximal ideal in this ring. The ring A_U of presheaf sections over an open set $U \subset$ Spec A is a ring of functions defined at every point of U. (If U is a distinguished open set, A_U is the ring of all such functions.) The kernel of a restriction map $\rho_{U,V}: A_V \to A_U$ (where $U \subset V$) is the set of functions which vanish at every point of U, though they may not vanish on all of V. The elements of A_U which are not restrictions of functions in A_V are inverses of functions which have zeros in $V - U$.

The topology of Spec A induces a topology in E, the weakest one in which the identification map from E to Spec A is continuous. In this topology, the closure of a point $x \in E$ is the set of all points in E which satisfy the same polynomials as x.

We choose a particular ring A to study in more detail. In $Q[x, y]$, let $I = (xy)$, and

$$A = Q[x, y]/(xy) \simeq Q[\bar{x}, \bar{y}],$$

where $\bar{x}\bar{y} = xy + (xy) = (xy) = 0$. Then

$$E = \{(z, w) \in C^2 \mid zw = 0\} = \{(z, 0) \mid z \in C\} \cup \{(0, w) \mid w \in C\},$$

the union of the complex coordinate axes (planes) in C^2.

The ideals $(\bar{x})$ and $(\bar{y})$ in A are prime but not maximal, since $Q[\bar{x}, \bar{y}]/(\bar{x}) \simeq Q[\bar{y}] \simeq Q[y]$, which is not a field. The points (z, w) in E whose corresponding ideal is $(\bar{x})$ are those of the form $(0, w)$ where w is transcendental over Q, and similarly, the points corresponding to the ideal $(\bar{y})$ are of the form $(z, 0)$ for transcendental z, since the homomorphism of $Q[\bar{x}, \bar{y}] \to C$ given by $\bar{x} \to 0$, $\bar{y} \to w$ has kernel $(\bar{x})$ if and only if w is transcendental. Further, $(\bar{x}) \cap (\bar{y}) = (0)$ in A, any prime ideal contains either $(\bar{x})$ or $(\bar{y})$, and the set of zero divisors in A is $(\bar{x}) \cup (\bar{y})$.

The maximal prime ideals in A are kernels of homomorphisms $\phi: [\bar{x}, \bar{y}] \to C$ for which the image is a field, so that $\phi(\bar{x})$ and $\phi(\bar{y})$ must be algebraic, and since $\phi(\bar{x})\phi(\bar{y}) = \phi(\bar{x}\bar{y}) = 0$, one of $\phi(\bar{x})$ and $\phi(\bar{y})$ must be zero, and the other algebraic. Such points $(\phi(\bar{x}), \phi(\bar{y}))$ in E are either of the form $(z, 0)$ with z algebraic, or $(0, w)$ with w algebraic. The kernels of the homomorphisms to

$(z, 0)$ and $(0, w)$ are the ideals $(p(\bar{x}), \bar{y})$ and $((\bar{x}), q(\bar{y}))$ where p is the minimal polynomial of z (and q of w). These ideals together with $(\bar{x})$ and $(\bar{y})$ are precisely the points of Spec A.

The points of $E \subset C^2$ are divided into equivalence classes corresponding to the ideals in Spec A: to each of (x) and (y) there corresponds a class of transcendental points, while to each maximal ideal $(p(\bar{x}), \bar{y})$ (where p is irreducible) there corresponds the set of algebraic points $\{(z, 0)\}$ where z is a root of p.

In the topology induced on E by Spec A, the closure of a transcendental point $(z, 0)$ is the z-axis and the closure of an algebraic point $(z, 0)$ is all $(z', 0)$ where z' is also a root of the minimal polynomial of z. A basis for the open sets in Spec A is the collection of all sets of the form $V_p = \{P \in \text{Spec } A \mid p \notin P\}$. In the special case $p = \bar{x}$,

$$V_{\bar{x}} = \{P \in \text{Spec } A \mid \bar{x} \notin P\} = \{(\bar{y}, p(\bar{x})) \mid p \in A \text{ is irreducible or zero, } p \neq \bar{x}\}.$$

The open set $V_{\bar{x}} \subset \text{Spec } A$ corresponds in E to the complement of the w-axis. Similarly, $V_{\bar{y}}$ may be regarded as the complement of the z-axis. The complement of a finite set of algebraic points corresponding to the maximal ideals

$$(\bar{x}, p_1(\bar{y})), \cdots, (\bar{x}, p_k(\bar{y})), \quad (p_{k+1}(\bar{x}), \bar{y}), \cdots, (p_n(\bar{x}), \bar{y})$$

is $V_{p_1(\bar{y})} \cap \cdots \cap V_{p_n(\bar{x})}$. Since every polynomial $p \in A = Q[\bar{x}, \bar{y}]$ can be written as $p = p_1(x) + p_2(y) - c$, where $p_1(0) = p_2(0) = p(0, 0) = c$, the points in V_p are $(z, 0)$ where $p_1(z) = 0$, and $(0, w)$ where $p_2(w) = 0$.

Geometrically, the ring $A = Q[\bar{x}, \bar{y}]$ is the ring of polynomials defined everywhere on E, the union of the complex axes. At a point (z, w) in E (where either z or w is 0) corresponding to a prime ideal P, the local ring A_P contains all rational functions whose denominators do not vanish at (z, w). The local ring $A_{\bar{x}}$ is the local ring at any $(0, w)$ where w is transcendental. Algebraically, we have the set of denominators $S = A - (\bar{x}) = \{p \mid p_2(\bar{y}) \neq 0\}$ where, as above, $p = p_1 + p_2 - c$. The kernel I of the homomorphism $\alpha: A \to A_{\bar{x}}$ is $\{q \in A \mid \exists p \in S$ such that $pq = 0\}$. If $p_1 + p_2 - c$ is a zero divisor, $c = 0$; if such a point is also in S, we have $p_2(\bar{y}) \neq 0$ and $p_2(0) = 0$, so it is an element of $(\bar{y}) - \{0\}$. Therefore $I = (\bar{x})$. Thus $A/I = Q[\bar{x}, \bar{y}]/(\bar{x}) = Q[\bar{y}]$ and the image of S in $Q[\bar{y}]$ is $Q[\bar{y}] - \{0\}$; so $A_{\bar{x}}$ is isomorphic to the field of rational functions in y. Similarly $A_{\bar{y}} = Q(x)$.

For the local ring at a maximal prime other than $(\bar{x}, \bar{y})$, for instance $P = (\bar{x}, \bar{y}^2 - 2)$, a similar argument will lead to the ring $A_P = \{p/q \in Q(y) \mid (y^2 - 2)$ does not divide $q\}$. That is, A_P contains inverses for all functions that do not vanish at $(0, \sqrt{2})$, while two polynomials in $\bar{x}$ and $\bar{y}$ which agree on an open set containing $(0, \sqrt{2})$ are identified in A_P. Thus the map $\alpha: A \to A_P$ is neither one-to-one nor onto. Now if $P = (\bar{x}, \bar{y})$, the set of denominators for A_P is exactly the set of polynomials $p_1 + p_2 - c$ for $c \neq 0$. So in this case the map $\alpha: A \to A_P$ is an inclusion which is not onto.

Finally, we describe the rings of presheaf sections A_U, for open sets U. If U is the complement of a finite closed set of maximal primes $(\bar{x}, p_1(\bar{y})), \cdots,$

$(\bar{x}, p_k(\bar{y}))$, $(p_{k+1}(\bar{x}), \bar{y})$, $\cdots$, $(p_n(\bar{x}), \bar{y})$ the set of denominators for A_U is $\{p\in A \mid p \notin \bigcup_{P\in U} P\}$, which is the set of all products $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with nonzero constant term. The map from $A_U \to A_P$ when $P\in U$ will have kernel $(\bar{x})$ if $P=(\bar{x}, p(\bar{y}))$ (where p is any irreducible polynomial except $\bar{y}$); $(\bar{y})$ if $P=(\bar{y})$ or $(\bar{y}, p(\bar{x}))$ (where $p(x)\neq x$); and (0) if $P=(\bar{x}, \bar{y})$.

If $U=V_{\bar{x}}$ is the complement of the w-axis, the set of denominators for A_U is the set of polynomials $\bar{y}p(\bar{y})+c\bar{x}^n$, for $p\in Q[\bar{y}]$ and $n\geqq 0$. Thus

$$A_U = \{p(x)/x^n \mid p \in Q[x]\}.$$

Since $A_{(\bar{y})}\simeq Q(x)$, the map $\rho_{(\bar{y}),U}: A_U \to A_{(\bar{y})}$ from the ring of sections to the stalk at $(\bar{y})$ is one-to-one but not onto. For a maximal prime $(\bar{y}, p)\in V_{\bar{x}}$, where $p(x)\neq x$, $\rho_{(\bar{y},p),\,U}: A_U\to A_{(\bar{y},p)}$ is an inclusion since x does not divide p. If U is the complement of the closed set $(\bar{x}, p_1(\bar{y}))$, $\cdots$, $(\bar{x}, p_k(\bar{y}))$, then $V_{\bar{x}}\subset U$, and the restriction map $\rho_{V_{\bar{x}},U}: A_U\to A_{V_{\bar{x}}}$ has kernel $(\bar{y})$.

The ring of sections over Spec A is A, for every element of A vanishes at some point in Spec A. Thus for every kind of open set U in Spec A, the elements of the rings A_U are the rational functions defined at every point of U.

In particular the functions $\bar{x}$ and $\bar{x}+\bar{y}$ are sections on the open set Spec A. They agree on the proper open subset $V_{\bar{x}}$, for $(\bar{y})$ is the kernel of the restriction map $\rho_{V_{\bar{x}},\,\mathrm{Spec}\,A}$, but do not agree on $(\bar{x}, \bar{y})$, which is in the closure of $V_{\bar{x}}$. Thus the points $\bar{x}$ and $\bar{x}+\bar{y}$ in $A_{(\bar{x},\bar{y})}$, though distinct, cannot be separated by sections since any open set containing $(\bar{x}, \bar{y})$ must intersect $V_{\bar{x}}$. Thus this sheaf fails to be Hausdorff both vertically as well as horizontally.

3. The sheaf of differential forms. Let $p\in X$ where X is a C^∞ manifold over R^n. We denote by C_p^∞ the set of all functions from X to the real line R^1 which are C^∞ in some neighborhood of p. Clearly C_p^∞ is a vector space over R in which the sum $f+g$ is defined on the intersection of the domains of f and g.

A *tangent* to X at p is a linear function $t: C_p^\infty\to R^1$ such that

$$t(fg) = t(g)\cdot g(p) + f(p)\cdot t(g) \qquad \text{whenever } f, g \in C_p^\infty.$$

The set of all tangents to X at p forms a vector space over R which we call X_p, the *tangent space to X at p*. If $\gamma: [0, 1]\to X$ is a C^∞ function such that $\gamma(t_0)=p$, and if $f\in C_p^\infty$, then $f\circ\gamma: [0, 1]\to X\to R^1$. If $(f\circ\gamma)'$ is the derivative of $f\circ\gamma$, the function $\gamma_*: C_p^\infty\to R^1$ defined by $\gamma_*(f)=(f\circ\gamma)'(t_0)$ is a tangent. To interpret this geometrically, we consider a particular local coordinate system (U_p, π_p) where $\pi_p: U_p\to R^n$. If $\pi_i: R^n\to R^1$ is the projection function defined by $\pi_i(t_1, \cdots, t_n)=t_i$, we write $x_i=\pi_i\circ\pi_p$; then $\pi_p(p)=(x_1(p), \cdots, x_n(p))\in R^n$. If $e_i=(0, 0, \cdots, 1, \cdots, 0)\in R^n$, we think of the curve $\gamma(t)=\pi_p^{-1}(\pi_p(p)+te_i)$ as the ith coordinate axis in U_p since $(d/dt)(\pi_p\circ\gamma)\big|_0=e_i$. Then the tangent γ_* satisfies

$$\gamma_*(f) = (f\circ\gamma)'(0) = \frac{d}{dt}(f\circ\pi_p^{-1}\circ\pi_p\circ\gamma)\bigg|_0$$

$$=\left(\frac{\partial(f\circ\pi_p^{-1})}{\partial t_1},\cdots,\frac{\partial(f\circ\pi_p^{-1})}{\partial t_n}\right)\Bigg|_{\pi_p(p)}\cdot e_i=\frac{\partial(f\circ\pi_p^{-1})}{\partial t_i}\Bigg|_{\pi_p(p)}.$$

Thus $\gamma_*(f)$ is usually denoted by $(\partial/\partial x_i)f$; the tangents $(\partial/\partial x_i), \cdots, (\partial/\partial x_n)$ form a basis for the vector space X_p [8, p. 7]. We think of X_p as an n-dimensional hyperplane tangent to X at p (Fig. 9). If

$$\gamma_* = \sum a_i \frac{\partial}{\partial x_i},$$

then $\gamma_*(f)$ is $a\cdot\operatorname{grad} f$, that is, the derivative of f in the direction $a=(a_1, \cdots, a_n)\in R^n$.

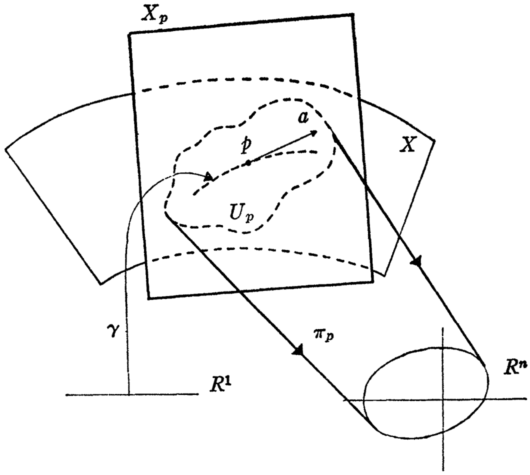

FIG. 9

If $\phi: X\to Y$ is a C^∞ mapping of manifolds we define the *differential of* ϕ to be the linear map $d\phi: X_p\to Y_{\phi(p)}$ defined as follows: if $t\in X_p$ and $f\in C^\infty_{\phi(p)}$ then $[d\phi(t)](f)=t(f\circ\phi)$. If (U_p, π_p) is a local coordinate system at $p\in X$ then $x_i=\pi_i\circ\pi_x$ defines a C^∞-map from the submanifold U_p to the manifold R^1. Hence the differential dx_i of x_i is a linear transformation from X_p to $R^1_{x_i(p)}$. Since R^1_t is isomorphic to R^1 for any $t\in R^1$, we may consider dx_i as an element of the dual vector space X_p^* of X_p. Since $dx_i\,(\partial/\partial x_j)=\delta_{ij}$ (where $\delta_{ij}=0$ if $i\neq j$, and 1 if $i=j$), the differentials $dx_i, \cdots, dx_n$ form a basis of X_p^* dual to the basis $\partial/\partial x_i, \cdots, \partial/\partial x_n$ of X_p.

Whereas each differential dx_i is a function of just one variable, a differential form in general is a function of several variables. To be precise, a *differential k-form* θ on X_p is an alternating k-linear function from k-tuples of elements of X_p to R^1. (θ is *k-linear* if it is linear in each variable separately, and *alternating* if $f(t_1\cdots t_k)=\operatorname{sgn}\sigma f(t_{\sigma(1)}\cdots t_{\sigma(k)})$ where $\sigma:\{1\cdots k\}\to\{1\cdots k\}$ is a permutation and $\operatorname{sgn}\sigma$ is $+1$ if σ is even, and -1 if σ is odd.) We denote by the *wedge product* $dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ the unique k-linear form on X_p defined on a basis for the set of k-tuples by

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}\left(\frac{\partial}{\partial x_{j_1}}, \cdots, \frac{\partial}{\partial x_{j_k}}\right) = \delta,$$

where $\delta = 1$ if $(i_1, \cdots, i_k) = (j_1, \cdots, j_k)$ and 0 otherwise. The set of all k-linear forms on X_p is an n^k dimensional vector space over R and the n^k forms $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where $i_j \in \{1, \cdots, n\}$, form a basis for this vector space. The set of differential k-forms is a subspace of the set of all k-linear forms and it has as a basis the k-linear forms $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $i_1 < i_2 < \cdots < i_k$. We call such a sequence of indices an *increasing k-tuple*. Thus the dimension of the space of differential k-forms is $\binom{n}{k}$ if $k \leqq n$ and 0 if $k > n$.

Let S_k denote the set of all increasing k-tuples of positive integers less than or equal to n. For $x \in S_k$ we denote by dx_s the k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where $s = (i_1, \cdots, i_k)$. Now let $U \subset X$ be an open set and θ a function which assigns to each $p \in U$ a differential k-form at p, $\theta(p)$. Thus, once having chosen a local coordinate system at p, so the forms dx_1, dx_2, $\cdots$, dx_n are defined, we may write

$$\theta(p) = \sum_{s \in S_k} a_s(p) dx_s,$$

where each $a_s(p)$ is the coefficient in the expansion of $\theta(p)$ in terms of the basis $\{dx_s\}_{s \in S_k}$. Thus a_s may be considered as a function from U to R^1. Since each $\theta(p)$ is a differential k-form at p, we call θ a C^∞ *k-form* on U if each function a_s is C^∞; we denote by $\Omega^k(U)$ the real vector space of all C^∞ k-forms on U. If $U \subset V$ then there is a linear transformation $\rho_{U,V}: \Omega^k(V) \to \Omega^k(U)$ defined by restriction of the domains of $\theta \in \Omega^k(V)$. The collection $\{\Omega^k(U)\}$ for U open in X together with the linear transformations $\rho_{U,V}$ is called the *presheaf of differential k-forms.*

Since every paracompact manifold has a C^∞ partition of unity subordinate to any open covering $\{U_\alpha\}$ [**8**, p. 85], presheaves over such manifolds satisfy the following special property: If $U = \bigcup U_\alpha$ where U_α is open in X, and if $\theta_\alpha \in \Omega^k(U_\alpha)$ are coherent in the sense that the restrictions to $U_\alpha \cap U_\beta$ of θ_α and θ_β agree (whenever $U_\alpha \cap U_\beta \neq \emptyset$) then there exists a unique $\theta \in \Omega^k(U)$ whose restriction to each U_α is θ_α. Certainly if $\{f_\alpha\}$ is a C^∞ partition of unity for $\{U_\alpha\}$ so that $f_\alpha: U \to R^1$, f_α vanishes off U_α, and $\sum f_\alpha = 1$; so we may define θ to be $\sum f_\alpha \theta_\alpha$.

Any such presheaf is called a sheaf, so when X is paracompact, we will call the system $\{\Omega^k(U), \rho_{U,V}\}$ the *sheaf of differential k-forms.* If $\Omega^k(p)$ is the set of differential k-forms at p, we may think of $\Omega^k(p)$ as the stalks of the sheaf S: $S = \bigcup_{p \in X} \Omega^k(p)$. The projection $\rho: S \to X$ assigns to each differential form at p the point p. The C^∞ k-form $\theta \in \Omega^k(U)$ is a section of S, and the collection $\{\theta(U) \subset S \mid U \text{ is open in } X\}$ forms a basis for the topology on S.

4. Sheaves: General definition. Each of the three previous examples reflects a different facet of the general concept of a sheaf. To emphasize this

	Sheaf of Germs of Holomorphic Functions	Sheaf of Local Rings	Sheaf of Differential Forms
Germ	$[f]_z$	$a/s \in A_P$	
Stalk Space	$S=\{(x,\ [f]_z)\}$	$S=\bigcup\{A_P \mid P\in \text{Spec } A\}$	
Projection	$\rho: S\to C^n$	$\rho: S\to \text{Spec } A$	
Stalk	$\{(z,\ [f]_z) \mid f\in A_z\}$	A_P(local ring)	
Base Space	C^n	Spec $A=\{P \mid P$ is prime ideal in $A\}$	$X=C^\infty$manifold
Presheaf Section		$a/s\in A_U$	$\theta\in\Omega^k(U)$
Restriction Homomorphism		$\rho_{U,V}: A_V\to A_U$	$\rho_{U,V}: \Omega^k(V)\to\Omega^k(U)$

FIG. 10

variety and to provide a coherent framework for the subsequent general definition, we summarize in Fig. 10 the three sheaves already discussed.

The sheaf of germs of holomorphic functions was defined to be the stalk space S together with a topology and a local homeomorphism ρ onto C^n. The sheaf of local rings was defined similarly, though in this case we also identified the system consisting of the rings A_U and the restriction homomorphisms $\rho_{U,V}: A_V\to A_U$ as a presheaf of rings. In the third case, the sheaf of differential k-forms was simply the presheaf of differential k-forms whenever the base space X was paracompact. The recognition of the equivalence of these two descriptions constitutes the beginning of sheaf theory. We now introduce definitions to formalize these two approaches and prove the definitions equivalent.

DEFINITION I. Let (X, τ) be a topological space, and let $\mathcal{C}$ be a class of similar mathematical objects (e.g., abelian groups, modules, rings). Let F be a function from τ to $\mathcal{C}$ and suppose for each pair U, $V\in\tau$ for which $U\subset V$ there is a map (e.g., a homomorphism, module homomorphism, or isomorphism) $\rho_{U,V}: F(V)\to F(U)$ which preserves the structure of the objects of $\mathcal{C}$. If $\rho_{U,V}\circ\rho_{V,W}=\rho_{U,W}$ whenever $U\subset V\subset W$, and if $\rho_{U,U}$ is the identity function on $F(U)$, we define the function F together with the *restriction* maps $\rho_{U,V}$ to be a *presheaf* over X. The elements of $F(U)$ are called *sections* of F over U. (In the language of category theory, a presheaf is a contravariant functor from the category of open sets and inclusion maps of X to some category of objects and morphisms.) A *sheaf* is a presheaf which satisfies the following two coherence axioms:

1. *If $\{U_\alpha\}$ is a family of open sets in X, if $U=\bigcup U_\alpha$, and if the section s, $t\in F(U)$ agree on each U_α (i.e., if $\rho_{U_\alpha,U}(s)=\rho_{U_\alpha,U}(t)$ for each α), then $s=t$.*

2. *If $\{U_\alpha\}$ is a family of open sets in X, if $U=\bigcup U_\alpha$,* and if *the sections $s_\alpha\in F(U_\alpha)$ are coherent in the sense that the restrictions of s_α and s_β to $U_\alpha\cap U_\beta$ agree (i.e., if $\rho_{U_\alpha\cap U_\beta,U_\alpha}(s_\alpha)=\rho_{U_\alpha\cap U_\beta,U_\beta}(s_\beta)$ whenever $U_\alpha\cap U_\beta\neq\varnothing$) then there exists a section $s\in F(U)$ such that for each α, $\rho_{U_\alpha,U}(s)=s_\alpha$.*

DEFINITION II. Let (X, τ) be a topological space, and let $\mathcal{C}$ be a class of similar mathematical objects (e.g., abelian groups, modules, rings). A *sheaf* over X is a triple $\{S, \pi, X\}$, where S is a topological space and $\pi: S \to X$ is a local homeomorphism (i.e., a map such that each point $p \in S$ has a neighborhood U on which $\pi|_U$ is a homeomorphism) such that each *stalk* $\pi^{-1}(x) \in \mathcal{C}$, and each operation is continuous as a function from $\bigcup_{x \in X}(\pi^{-1}(x) \times \pi^{-1}(x))$ (with the topology induced from $S \times S$) to $\bigcup_{x \in X} \pi^{-1}(x) = S$.

We shall call a sheaf of type I a *sheaf of sections*, and a sheaf of type II a *sheaf of germs*. The relationship between these two types of sheaves is precisely as illustrated by the preceding examples.

To be specific, suppose F is a presheaf (of sections) over W; we construct the corresponding sheaf of germs by defining a germ at $x \in X$ to be an equivalence class of $A_x = \bigcup_{x \in U} F(U)$ under the relation $s \in F(U) \sim t \in F(V)$ if $\rho_{W,U}(s) = \rho_{W,V}(t)$ for some $W \subset U \cap V$. Thus the germ of a section $s \in F(U)$ at a point $x \in U$ is the collection of all sections $t \in F(V)$ which agree with s on some neighborhood V of x. We denote, as usual, the germ of s at x by $[s]_x$, and let the stalk space S be $\{(x, [s]_x) | s \in F(U)$, where $x \in U\}$. The topology on S is generated by neighborhoods of the form $V(s, U) = \{(x, [s]_x | x \in U\}$, so the projection $\pi: S \to X$ becomes a local homeomorphism. By interpreting the sections as functions, the so-called restriction maps $\rho_{U,V}$ really are restrictions and the topology on S is the strongest relative to which the sections are continuous. (The topology on S can also be characterized as the quotient of the topology on $\bigcup_{U \in \tau} (U \times F(U))$ under the equivalence relation induced by $\sim$, where each U carries the subspace topology and $F(U)$ is discrete.)

Each stalk $\pi^{-1}(x)$ clearly inherits the operations of the $F(U)$ and each such operation is continuous. For example, if each $F(U)$ is an abelian group under addition and if $[s]_x$ and $[t]_x \in \pi^{-1}(x)$, then $[s]_x + [t]_x$ is defined to be $[s+t]_x$. If $V(s+t, U)$ is a neighborhood of $[s+t]_x$, then the inverse image of $V(s+t, U)$ under $+$ contains $\{(r, q) | r \in V(s, U), q \in V(t, U), \pi(r) = \pi(q)\}$, an open set in $\bigcup_{x \in X}(\pi^{-1}(x) \times \pi^{-1}(x))$. Thus $\{S, \pi, X\}$ is indeed a sheaf of germs.

Conversely, suppose $\{S, \pi, X\}$ is a sheaf of germs (perhaps one constructed as above from some presheaf). If U is open in X, let $F(U)$ be the collection of continuous functions $s: U \to F$ such that $\pi \circ s$ is the identity on U. $F(U)$ inherits the algebraic structure from the stalks by pointwise definitions, and the restriction maps $\rho_{U,V}$ are just that—the restriction of s from V to the subset U. The first coherence axiom for sheaves is satisfied trivially since the sections are functions, and the second is satisfied since the $F(U)$ contain all continuous functions from U to F which are inverses of π.

Now if $\{F, \pi, X\}$ is a sheaf of germs, the sheaf derived from its (pre)sheaf of sections is canonically isomorphic to F. However, if S is a presheaf of sections, the presheaf of sections S' associated with the sheaf of germs derived from S is generally different from S, for since S' is a sheaf, it may have more sections than S (in order to satisfy the second coherence axiom), while some sections which were distinct in S may be identified in S' (because of the first axiom). Of course, if S is a sheaf, then S' is naturally isomorphic to S, so in this sense the two definitions of a sheaf are essentially equivalent.

A common and convenient alternative to the construction of equivalence classes is the use of direct limits. If F is a presheaf of sections over X, and if $x \in X$, the restriction of F to the neighborhoods of x forms a *directed system* $(\{F(U)\}_{x\in U}, \{\rho_{U,V}\}_{x\in U\subset V})$ since $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$, whenever $x \in U \subset V \subset W$. A few elements of such a system may be represented by the commutative diagram of Fig. 11.

$$\begin{tikzcd}
F(W_1) \arrow[rr, "\rho_{U,W_1}"] \arrow[dr, "\rho_{V_1,W_1}"'] & & F(U) \\
& F(V_1) \arrow[ur, "\rho_{U,V_1}"'] \arrow[dr, "\rho_{V_1\cap V_2,V_1}"] & \\
F(W_2) \arrow[ur, "\rho_{V_1,W_2}"] \arrow[dr, "\rho_{V_2,W_2}"'] & & F(V_1\cap V_2) \\
& F(V_2) \arrow[ur, "\rho_{V_1\cap V_2,V_2}"'] &
\end{tikzcd}$$

FIG. 11

The direct limit of this system is, roughly speaking, the first object which can appear to the right of the diagram. Specifically, an object F_x together with maps $\rho_U: F(U) \to F_x$ for each $F(U)$ is called a *direct limit* of the system $(\{F(U)\}_{x\in U}, \{\rho_{U,V}\}_{x\in U\subset V})$ provided that

(i) whenever $U \subset V$ the diagram in Fig. 12 commutes,

(ii) F_x is universal with respect to property (i)—that is, if $(G_x, \{\sigma_y\})$ also satisfies property (i), then there exists a unique map $\eta: F_x \to G_x$ such that for each $\sigma_U: F(U) \to G_x$, $\sigma_U = \eta \circ \rho_U$.

$$\begin{tikzcd}
F(V) \arrow[rr, "\rho_V"] \arrow[dr, "\rho_{U,V}"'] & & F_x \\
& F(U) \arrow[ur, "\rho_U"'] &
\end{tikzcd}$$

FIG. 12

Condition (i) makes explicit the idea of "appearing on the right of the diagram" while condition (ii) asserts that F_x is the first such object. It follows trivially from these conditions that the direct limit is unique (up to isomorphism), so we denote it by $\varinjlim_{x\in U} F(U)$.

Consider now the stalk of germs at x derived from the presheaf F. If ρ_U denotes the map from $F(U)$ to the stalk $\pi^{-1}(x)$ defined by $\rho_U(s) = (x, [s]_x)$ (where $x \in U$), then $\pi^{-1}(x) = \varinjlim_{x\in U} F(U)$ since whenever $x \in U \subset V$, $\rho_V = \rho_U \circ \rho_{U,V}$ and $\pi^{-1}(x)$ is universal with respect to that property. To prove this last assertion we assume that $(G, \{\sigma_U\})$ is another direct limit and observe that if both

$t_1 \in F(U_1)$ and $t_2 \in F(U_2)$ are in the same germ $[s]_x$, then $\sigma_{U_1}(t_1) = \sigma_{U_2}(t_2)$, for by the definition of $[s]_x$, there exists some $V \subset U_1 \cap U_2$ such that $\rho_{V,U_1}(t_1) = \rho_{V,U_2}(t_2)$. Then by the commutativity of the diagram in Fig. 13, we have

$$\sigma_{U_1}(t_1) = \sigma_V \circ \rho_{V,U_1}(t_1) = \sigma_V \circ \rho_{V,U_2}(t_2) = \sigma_{U_2}(t_2).$$

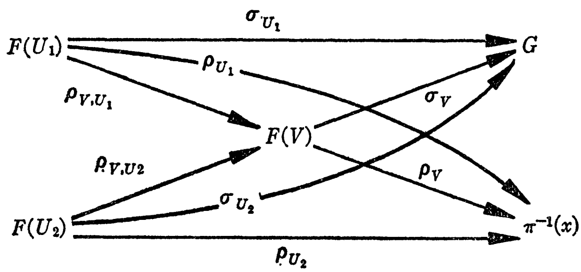

FIG. 13

Thus each element $[s]_x$ of $\pi^{-1}(x)$ is mapped to some point of G by the appropriate $\sigma_U \circ \rho_U^{-1}$: we call this point $\eta(x, [s]_x)$ and thereby define the required unique map from $\pi^{-1}(x)$ to G.

So we may summarize all three approaches in one sweeping generalization: for each $x \in X$, the stalk over x of the sheaf of germs is the direct limit of the restriction of the presheaf to the neighborhoods of x.

We close this section with a fourth characterization of sheaves, this one also based on a universal property, and illustrated to some extent in the previous examples. Suppose $\mathfrak{F}$ is a class of functions defined on open subsets of a topological space X. If $F(U)$ is the collection of all $f \in \mathfrak{F}$ whose domain is U, and if $\rho_{U,V}$ is the restriction map (i.e., $\rho_{U,V}(f) = f|_U$ whenever $U \subset V$ and $f \in F(V)$), the collection $\{F(U), \rho_{U,V}\}$ is a presheaf. If this presheaf is a sheaf S (as it will be if $\mathfrak{F}$ is the set of holomorphic functions on open subsets of C^n, or the set of differential forms on open subsets of a paracompact manifold X) then we can define a universal continuous function $\Phi: S \to R$ of type $\mathfrak{F}$ so that each $f \in F(U)$ factors uniquely through the sheaf S: that is, there exists a unique $\hat{f}: U \to S$ such that $\hat{f} = \Phi \circ f$.

In general, any object S and map $\Phi: S \to R$ with the property that each function $f: U \to R$ factors uniquely through S via Φ is called universal with respect to the characterizing properties of the functions f. The pair (S, Φ) is uniquely determined (up to isomorphism) by this property. Thus, for instance, the sheaf of holomorphic functions is characterized by being the unique universal object for the family of holomorphic functions.

Since $\hat{f}^{-1}(V(f, U)) = U$, $\hat{f}$ is continuous. Thus sheaves transform a complicated property of functions, such as analyticity, into the simpler one of continuity, for the topology on the sheaf is chosen precisely so that a continuous section on U (i.e., an element of $F(U)$) corresponds to one of the specialized (e.g., analytic, differentiable) functions of $F(U)$.

5. History and applications. Sheaf theory is a particularly effective tool in those areas which ask for global solutions to problems whose hypotheses are local. Among the early papers which introduced the ideas, though not the language, of sheaf theory, many were concerned with the Cousin problems from the theory of functions of several complex variables; the first (or additive) and second Cousin problems ask respectively about the existence of a meromorphic function with specified poles and the existence of a holomorphic function with specified zeros. Henri Cartan and Kiyoshi Oka independently solved these problems, working in the ring of germs of holomorphic functions introduced in our first example, where the operations take into account the domains of the functions. Oka [1950] cites Cartan [1940] as the source of the notion of "*idéal holomorphe de domaines indéterminés*" in this ring, and both Oka [1951] and Cartan [1944] refer to the article of W. Rückert [1933] which took the concept of ideal from polynomial rings and interpreted it in the ring of functions on a fixed domain. Cartan [1944] carried on the investigation of the sheaf of germs, still in the earlier terminology, clarifying the relations among the problems without achieving solutions.

Independently, Oka in 1948 wrote a paper [1950] (seventh in a series published from 1936 to 1953 and collected in a single volume [1961]) which developed the same material in a more complete form, and carried it through to a solution of the first Cousin problem. Building on Oka's paper, Cartan was able to solve the second problem as well as to simplify Oka's solution to the first, and his paper [1950] and Oka's were published together. A footnote acknowledges Oka's solution of the second problem in the meantime [1951].

The 1950 Cartan paper for the first time phrases the questions in the sheaf theoretic terms which had been developed in the *Séminaire Cartan* in 1948–49. An analytic sheaf, that is a sheaf of modules over the sheaf of germs of holomorphic functions, is called *coherent* over an open set U if for every $x \in U$ there is an open set U_x such that the sections over U_x generate the stalk at y for all y in a sufficiently small neighborhood of x. If $f_1, \cdots, f_k$ are functions holomorphic on a domain D, we may define the sheaf R of relations among the f_i by taking the sections R_U of R over U open in D to be the set of k-tuples of holomorphic functions $(g_1, \cdots, g_k)$ for which $\sum_{i=1}^{k} f_i g_i \equiv 0$ on U. In this vocabulary the first Cousin problem is to show that R is coherent, while the second problem similarly asks whether the sheaf over an analytic variety is coherent, where a variety is the set of common zeros of a set of holomorphic functions, and the ideal of sections over an open set is the ideal of functions on the variety which vanish on that open set.

Cartan borrowed the term "faisceau" (sheaf) from Leray [1945; 1946]. Leray's concept was closer to that of a "presheaf." Cartan [1953] attributes the topological definition to an exposition by Lazard in the *Séminaire Cartan* [1950]. In each case, the key concept was that of a system of local coefficients. Studying sets of invariants for an object (base space) by investigating what functions can be defined from it to some convenient object called a set of coefficients, as is done in cohomology, leads very naturally to a sheaf of coefficients since the presheaf structure allows coefficients to be assigned locally, that is, to each open subset of the base space. Formally, the principal construction of cohomology with coefficients in a (pre)sheaf follows the Čech construction of cohomology with fixed coefficients.

Let X be a topological space and S a sheaf over X, say of abelian groups. For any open cover $\mathfrak{U}$ of X, a *q-cochain*, q being a nonnegative integer, is an alternating function which assigns to every $q+1$-tuple of sets in the cover $\mathfrak{U}$ a section over the intersection of these sets (the zero section if the intersection is empty). $C^q(\mathfrak{U}, S)$ denotes the group of q-cochains. For each q a *coboundary operator* δ^q, $\delta^q: C^q(\mathfrak{U}, S) \to C^{q+1}(\mathfrak{U}, S)$ is defined by

$$\delta^q f(U_{i_0}, \cdots, U_{i_{q+1}}) = \sum_{j=0}^{q+1} (-1)^j f(U_{i_0}, \cdots, \hat{U}_{i_j}, \cdots, U_{i_{q+1}}),$$

where the caret over U_{i_j} means that U_{i_j} is to be omitted from the arguments of f, and each of the sections on the right is to be interpreted as restricted to the intersection of all the U_{i_j}. By convention we write δ for all δ^q. Since f is alternating, $\delta\delta = 0$, so the image $\delta(C^{q-1}(\mathfrak{U}, S))$, whose elements are called *coboundaries*, is contained not merely in $C^q(\mathfrak{U}, S)$, but in the set of *cocycles* $Z^q(\mathfrak{U}, S)$, the kernel of

$$\delta: C^q(\mathfrak{U}, S) \to C^{q+1}(\mathfrak{U}, S).$$

The qth *cohomology group* $H^q(\mathfrak{U}, S)$ of the cover $\mathfrak{U}$ with coefficients in S is the quotient group $Z^q(\mathfrak{U}, S)/\delta(C^{q-1}(\mathfrak{U}, S))$. Although the construction of the cohomology group uses only the presheaf of sections of S, the sheaf property allows us to interpret $H^0(\mathfrak{U}, S)$. In order for 0-cochain to be a cocycle,

$$\delta f(U_0, U_1) = \rho_{U_0 \cap U_1, U_1}(f(U_1)) - \rho_{U_0 \cap U_1, U_0}(f(U_0))$$

must be zero, and in any sheaf, a collection of sections so related defines a unique global section. Thus $H^0(\mathfrak{U}, S)$ is S_X—independent of the cover $\mathfrak{U}$. For all q, if $\mathfrak{U}$ is a covering which refines a covering $\mathfrak{V}$, the restriction maps can be used to define a canonical map $H^q(\mathfrak{V}, S) \to H^q(\mathfrak{U}, S)$. The direct limit, over all coverings $\mathfrak{U}$ of X, of the groups $H^q(\mathfrak{U}, S)$ with these maps, is the qth *cohomology group* of X with coefficients in the sheaf S and is denoted by $H^q(X, S)$.

One property frequently taken as axiomatic for cohomology theories holds also for this one. If $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$ is an exact sequence of sheaves over X, there is a *long exact sequence* of cohomology

$$\begin{aligned} 0 &\to H^0(X, F) \to H^0(X, G) \to H^0(X, H) \to H^1(X, F) \to \cdots \\ &\to H^{q-1}(X, H) \to H^q(X, F) \to H^q(X, G) \to H^q(X, H) \to H^{q+1}(X, F) \to \cdots \end{aligned}$$

[2, p. 28]. As usual, a pair of homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact* at B if $\operatorname{Im} f = \operatorname{Ker} g$, so the exactness of $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$ means that F is a subsheaf of G (this requires f to be an open mapping), g is onto, and H is isomorphic to the quotient sheaf G/F. The existence of the long exact sequence is the major reason for the usefulness of cohomology, for the 0-dimensional groups which begin the sequence are the groups of global sections, while the higher groups by their construction reflect the local properties of X.

For instance, if we take F to be the sheaf of germs of holomorphic functions on a complex manifold X, and G to be the sheaf of germs of meromorphic functions, then F is a subsheaf of G, and the global sections of the quotient sheaf G/F can be interpreted as the data of the first Cousin problem, since each section describes the behavior of a function near its poles [9, p. 161]. Thus, since this problem asks whether there exists a function meromorphic on X with such poles, the first Cousin problem may be interpreted as asking whether the last map in the sequence $0 \to F_X \to G_X \to (G/F)_X$ is onto. This sequence is the beginning of the long exact cohomology sequence, and the next group in that sequence is $H^1(X, F)$. The Cartan-Oka result is that $H^q(X, F)=0$ for all $q \geqq 1$ if X is a Stein manifold, a class of manifolds with "sufficiently many" holomorphic functions, which includes all Riemann surfaces which are connected and non-compact. In addition to proving this result, Cartan [1953] and Serre [1953] give other applications of the fundamental theorems for a Stein manifold X:

THEOREM A. *For every coherent analytic sheaf S over X, $H^0(X, S)$, which is the module of global cross sections S_X, generates the stalk S_x for every $x \in X$.*

THEOREM B. *For every coherent analytic sheaf S over X and $q \geqq 1$,*

$$H^q(X, S) = 0.$$

Properties A and B characterize Stein manifolds.

The proceedings of the 1954 AMS summer institute [1956] illustrate that by then the basic concepts of sheaf theory had been clarified apart from the original example, and the bibliographies indicate that applications had begun to diversify, particularly into algebraic geometry. For example, Kodaira and Spencer showed the equivalence of several different definitions of the arithmetic genus of an algebraic variety and provided a classification of complex line bundles [1953]. Hirzebruch gave a sheaf-theoretic statement and proof of the Riemann-Roch theorem [1953; 1956], and Weil of the deRham theorem [1952].

One seminar at the 1954 institute was based on an early version of Serre's major article FAC [1955], the first entirely algebraic development of sheaf theory. The applications to complex variables had frequently made use of complex integration, and this tool was not available in abstract algebraic geometry. "Faisceaux algébriques cohérents" are coherent sheaves which are sheaves of modules over the sheaf of local rings on an algebraic variety. Serre showed that if the base field is the field of complex numbers, the theory of algebraic coherent sheaves is isomorphic to the theory of analytic coherent sheaves. Going further in the direction of an algebraic treatment, Grothendieck dealt with sheaves in the context of cohomology in an abelian category [1957].

The publication of Godement's book [1958] signals the appearance of sheaf theory as an independent discipline. The bibliography which follows lists both recent treatments of sheaf theory and books on other subjects which make use of sheaves.

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ON THE FOUNDATIONS OF SET THEORY

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I want to discuss here the relevance to mathematicians, as teachers and researchers, of some of the recent discoveries about axiomatic set theory. Most readers have heard of these advances, which began just a few years ago with Cohen's work. The results are certainly intellectually amazing to all of us. I think they may even give rise to certain changes in our teaching and research, and the purpose of this paper is to describe some possibilities along these lines. To set the stage and fix the ideas I shall first describe a few of these discoveries in a fairly precise way. Then, in the nonexact portion of the paper, I shall discuss some possible changes in teaching and research, and also some philosophical views which are affected by these discoveries.

1. A survey of results. A much more comprehensive (and more technical) survey can be found in Mathias [7]. Here I state just a very few results, but I wish to emphasize that the nonmathematical arguments of the next section apply in some form to virtually all of the results described in [7]. I assume that the reader has a modest acquaintance with the idea of a language and a metalanguage, and with the precise notions of a (first-order) sentence, a (formal) proof, and a theorem. In this section I work in a metalanguage and talk about the language of mathematics. I leave the metalanguage unspecified in detail; to begin with I assume that it is rather weak, with just enough machinery to

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