UNSOLVED PROBLEMS IN GEOMETRY

The difficult problems we do tomorrow; the impossible ones take a little longer.

By **LYNN ARTHUR STEEN**
Saint Olaf College
Northfield, MN 55057

Draw a wiggly curve on a sheet of paper, without lifting your pencil or allowing the curve to cross itself, joining the end to the point of beginning. Now try to find four points on your curve that form the vertices of a square.

That problem in plane geometry appears to be a little bit harder but not radically different from the well-known "construction" problems of high school geometry courses: "Given a circle in the plane, construct an inscribed square." Our problem just has a few more wiggles in it.

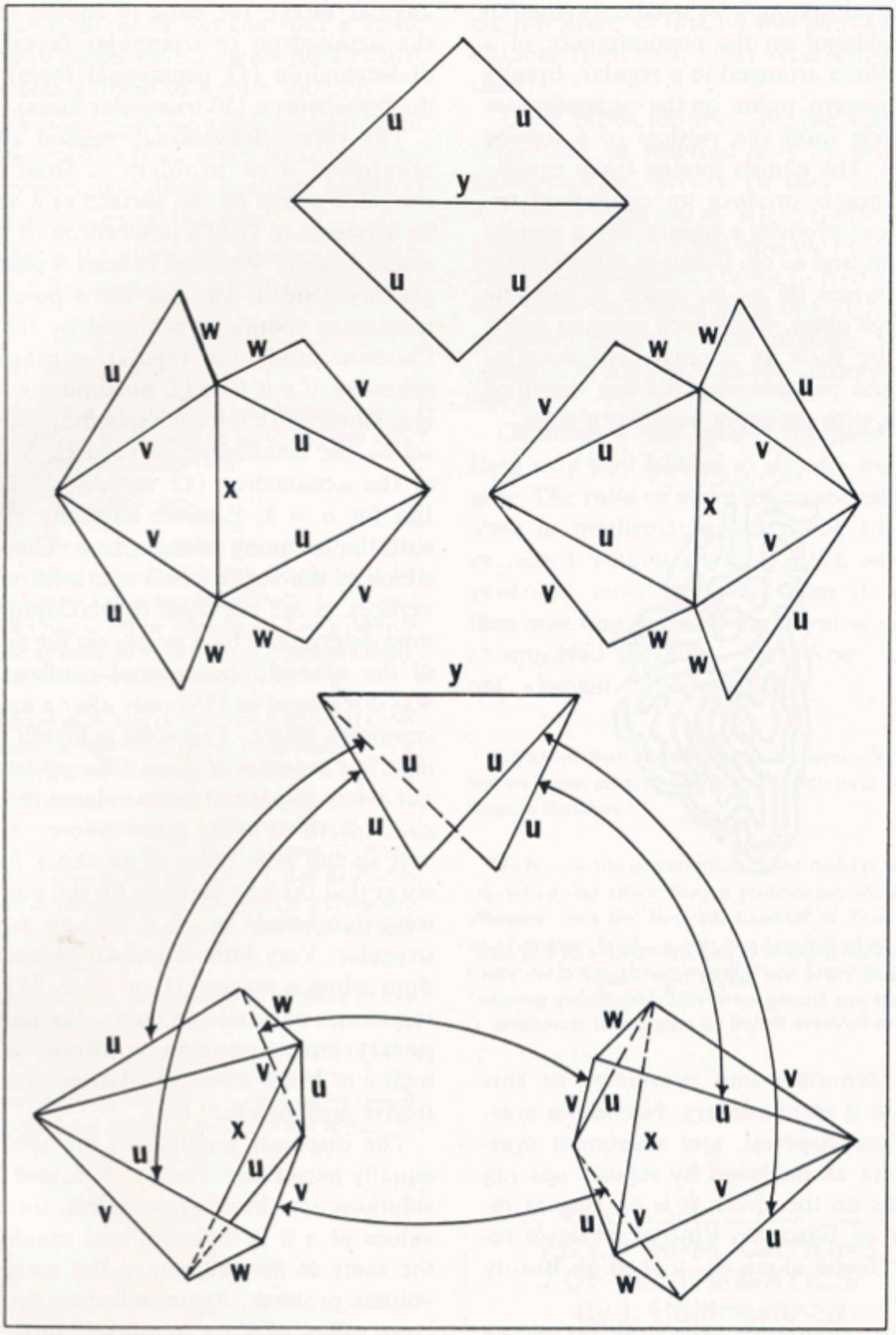
It also happens to be unsolved: No one has yet been able to prove that every closed curve contains the vertices of a square. Geometry, despite its procrustean image as an ancient, completed subject, abounds in unsolved problems that are still under active investigation. Many of these problems are easy to understand, and some of them are even being solved.

The most famous of the long-unsolved problems of geometry is the four-color conjecture that every map can be colored with no more than four colors in such a way that adjacent regions are assigned different colors. This conjecture, formulated in 1852, was first solved in 1976 with an innovative computer-assisted proof by Kenneth Appel and Wolfgang Haken of the University of Illinois (*Science News*: 7/31/76, p. 71).

Recently another major geometric question dating back to the early nineteenth century was resolved, but with a counterexample rather than a proof. Just last year Robert Connelly of Cornell University showed that something that had been believed for more than 150 years was actually false: that any closed polyhedral surface in three-dimensional space is rigid. A complete spherical geodesic dome is a common example of this type of surface. Although standard geodesic domes are rigid (because they are convex), Connelly was able to construct from 18 rigid triangles an enclosed surface that is flexible! Subsequently, this surface was simplified (see illustration) to the extent that a cardboard model can easily be constructed.

Connelly's counterexample to the "rigidity conjecture" is particularly striking because it was so unexpected. For centuries engineers have believed that triangular structures are rigid, and no mathematician said nay. Yet now any schoolchild can build a simple closed triangular surface that flexes. Of course, these special examples do not mean that *all* polygonal structures flex: The common ones, including those used in buildings and bridges, are indeed rigid. But these examples of flexible polyhedra do show that our confidence in the rigidity of structures must be based on more complex criteria than the simple beliefs of the past.

It is not uncommon for common sense to produce nonsense, especially when it is torn between the aesthetic demands of symmetry and the rational demands of logic. While symmetry and logic usually work in harmony, they occasionally produce jarring dissonance. Even the music of the spheres is sometimes disrupted by strange chords.



To build a flexible enclosed surface from triangular pieces, begin with two flat, butterfly-shaped pieces, each formed from six triangles. (The actual length of the edges is not critical, but the following choices work well: u = 12, v = 10, w = 5, x = 11, y = 17.) Push the upper flaps of the left butterfly down, and glue the w-length edges together; do the same for the lower flaps on the right butterfly. Then push the lower flaps of the left butterfly and the upper flaps of the right butterfly up, gluing the corresponding w-length edges together.

Now join the upper v-length edges of the two butterflies to each other, and, similarly, the lower ones to each other. This produces a contorted figure bounded by four u-length edges. Seal up the hole bounded by these edges by the folded rhombus with four edges of length u, as suggested in the diagram. The result is a (nonconvex) enclosed figure, formed entirely from triangles, that is not rigid.

For ideal harmony we need only listen to points playing on the circumference of a circle. When arranged in a regular, equally spaced pattern, points on the circumference of a circle form the vertices of a regular polygon: The chords joining three equally spaced points produce an equilateral triangle, four produce a square, five a regular pentagon, and so on. Points in these regular arrangements lie as far apart as possible from each other, while each polygon determined by such an arrangement contains the largest possible area for any inscribed polygon with the same number of sides.

Is it always possible to find the vertices of a square on a closed curve, no matter how wiggly it is? This problem is one among many from elementary geometry that so far have defied all attempts at resolution.

The economy and symmetry of this structure is extraordinary: Maximum area, maximum dispersal, and maximum symmetry are all achieved by regular spacing of points on the circle. It is an elegant reminder of Edna St. Vincent Millay's refrain: "Euclid alone has looked on Beauty bare."

In three dimensions, the circle becomes a sphere, and the regular polygons become regular solids called polyhedra. The regular polyhedra enjoy a distinctive harmony of their own, a happy coincidence of truth and beauty that has been held in awe since the time of ancient Greece. There are only five regular polyhedra, the so-called Platonic solids, namely the tetrahedron (4 triangular faces), the cube (6 square faces), the octahedron (8 triangular faces), the dodecahedron (12 pentagonal faces), and the icosahedron (20 triangular faces).

The three-dimensional version of the maximum area problem is this: How should n points on the surface of a sphere be arranged to yield a polyhedron of maximum volume? We need at least 4 points to get any solid at all, and for 4 points the maximum volume is achieved by the first Platonic solid—the regular tetrahedron. Likewise, if n is 6 or 12, maximum volume is achieved with the corresponding Platonic solid—the octahedron (6 vertices, 8 faces) or the icosahedron (12 vertices, 20 faces). But for $n = 8$, Platonic harmony clashes with the harmony of the sphere: The cube, which is the regular Platonic solid with 8 vertices, is *not* the solid of maximum volume determined by 8 points on the surface of the sphere! The optimal configuration was discovered in 1963 only after a massive computer search. This solid is highly irregular, having sides of three different lengths, but it has 12 percent more volume than the cube inscribed in the same sphere.

It should now come as no shock to discover that the best patterns for the non-Platonic numbers ($n = 5, 7, 9$, etc.) are equally irregular. Very little is known about solutions when n reaches 10 or more. Without symmetry as a reliable guide, the contemporary mathematician is almost at the mercy of brute force calculation—an ineffective and inelegant tool.

The dispersal problem of the sphere is equally perplexing. For $n = 4, 6$, and 12 its solutions are the Platonic solids; for other values of n it is different, and usually *not* the same as the solution to the maximum volume problem. Again, solutions for most large values of n are completely unknown.

Unsolved problems of geometry is the subject of a lengthy article by University of Washington geometer Victor Klee that appears in the May 1979 issue of *Mathematics Magazine*. Klee reports on the status of nearly a dozen major problems in plane geometry that have stood unsolved for decades. Here are some of them:

• Can a circle be broken into a finite number of pieces (like a jigsaw puzzle) that can be reassembled to form a square?

• Can every point on a polygonal billiard table be reached from every other point by an appropriate (albeit perhaps lengthy) shot that caroms off the sides?

• What is the minimum number of colors that can be used to color the entire Euclidean plane so that no two points at unit distance from each other receive the same color?

These problems may, like the four-color conjecture, be resolved only by extraordinarily complex proofs. Or they may yield to a very simple attack. Klee reports on one problem that remained unsolved for nearly 50 years that now is known to have a solution easily within the grasp of a high school geometry student! (This problem, and its solution, is contained in the accompanying box.)

Geometry's rich lode of unsolved problems may well be due to its very long history. The ratio of unsolved to solved problems in mathematics, according to Klee, increases without bound: Each advance generates more problems than it solves, thus ensuring a nearly exponential growth in unsolved problems, even from "classical" elementary geometry.

A HARD PROBLEM . . .

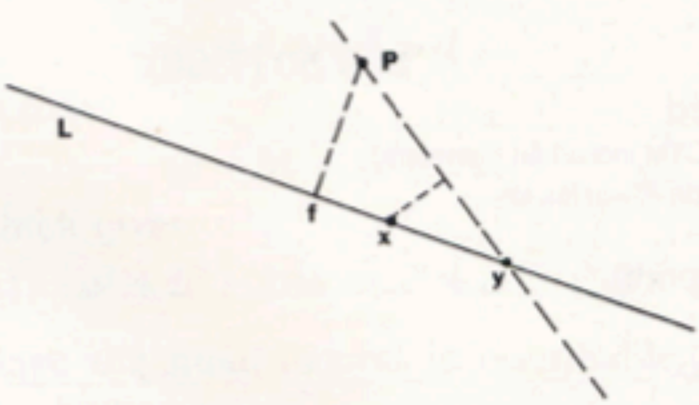
Suppose S is a finite set of points in the plane, not all collinear. Must there be some line that contains precisely two points of S?

. . . WITH AN EASY SOLUTION

This problem was posed near the end of the nineteenth century by the British mathematician James Joseph Sylvester, and remained unsolved for more than 40 years: After nearly half a century it looked as hard as it did when it was first posed. But then someone had a clever idea:

Pick a point of S, and draw a line that misses this point through two other points of S; measure the distance from the point you picked to the line you drew. Since there are only finitely many points in S, there can be only a finite number of lines through two points of S. Hunt among all these points and lines to find the pair p, L that produces the least distance between the point p and the line L. This line, as we can show, will then contain only two points of S.

Suppose the line L contained three or more points of S. Then two of them must lie on the same side of the foot f of the perpendicular from p to L. If x and y are two such points with x closer to f, the distance from x to the line through p and y would be less than the distance from p to the line L:



But this cannot be, since the distance from p to L is the least possible of all such distances between points and lines in S. Hence the line L cannot contain three or more points of S: It will, necessarily, contain precisely two points of S.

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