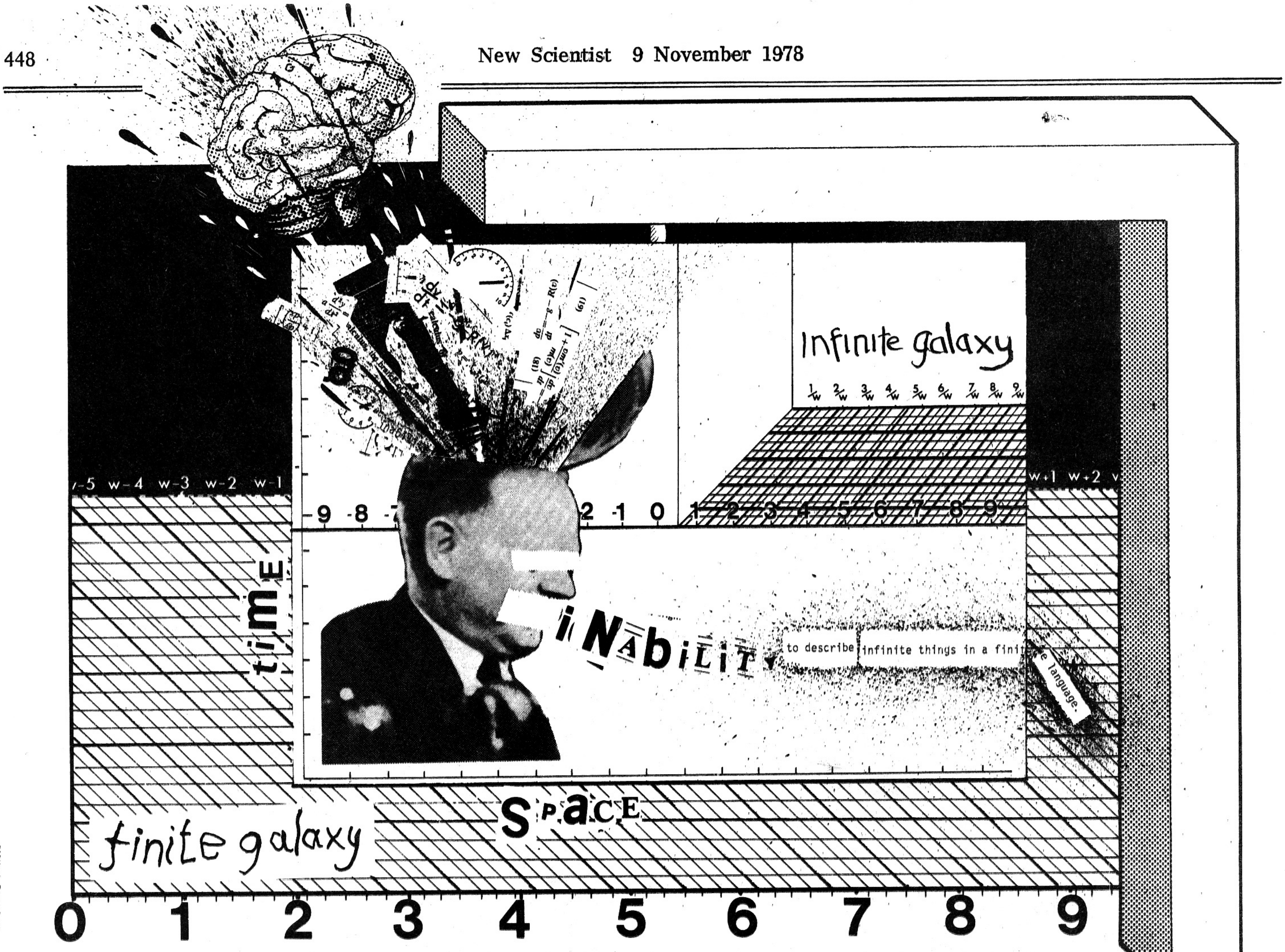

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A new perspective on infinity

Three centuries after Newton's introduction of infinitesimal calculus as a tool of mathematics, a new "non-standard" analysis provides a different view of the infinitely large and the infinitesimally small, and their relation to the finite world

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Man lives in a finite world, but his imagination does not. While our supply of coal and oil and even sunlight is limited, our supply of time and space is not. To understand a finite natural world, scientists create conceptual structures with infinitely many dimensions. Against the certainty of death, theologians speak of eternal life. The flights of man's fancy transcend the perceived reality of his world, conjuring visions of infinite structures far beyond the range of normal experience.

Infinite numbers and infinite sets inhabit a large but little-explored corner of the man-made universe of mathematical ideas. Because they are figments of man's imagination with no basis in observable fact, the infinitely large and infinitely small have puzzled mathematicians, scientists and philosophers for centuries. But in recent years an invisible revolution in the abstract world of higher mathematics has produced a totally new perspective on the concept of infinity. This new interpretation, created largely by the late Abraham Robinson of Yale University, promises to resolve one of the longest debates in scientific history and to provide a major new tool for both the pure and applied mathematical sciences.

By using esoteric techniques of mathematical logic, Robinson showed that the very reasoning processes that make mathematical thought possible must also make possible the existence of conceptual models that contain both infinitely large and infinitely small numbers. These new mathematical models, called "non-standard" models in contrast to the more traditional "standard" ones taught in university mathematics courses, have excited both mathematicians and scientists because they offer for the first time a real possibility of computing with infinite numbers.

Physicists and economists, for example, must often resort to manipulation of non-existent and sometimes undefinable infinite objects in order to get the results they need. Physicists use infinite volumes to express precisely certain subtle concepts of heat and electrical energy, while economists talk of idealised economies with infinitely many traders as a means of coping with the astonishing complexity of real but very large economies. Standard means of building these models rely on indirect constructions that are difficult to comprehend and foreign to even a sophisticated mathematical intuition. Non-standard models, however, provide direct access to the required infinite concepts, and do so in a manner that preserves the momentum of scientific intuition built up in simpler, better-understood finite models.

Ordinary economic models, for example, are usually based on an assumption that no individual agent has more than an infinitesimal influence on the course of the economy as a whole. No ordinary trader on the stock exchange will influence the price of the stock by his trade, yet the cumulative effect of a large number of individual trades does cause a change in price. This aggregation of insignificant effects is difficult to represent in mathematical form, and even more difficult to grasp in a useful, intuitive form. Non-standard methods, however, make it possible to represent the change in price as the result of infinitely many infinitesimal influences exerted by the individual trades.

Mathematicians too are excited by non-standard models because they reveal new and profound insight into the general process of building the abstract models studied in higher mathematics. They show, in particular, that the models and constructs with which mathematicians routinely work are not absolute and universal, but are intimately related to the nature and power of the language in which all mathematics is expressed. Mathematical models—from the whole numbers every child learns at school to advanced calculus and beyond—are, in some sense, just very powerful figures of speech. The medium moulds the message, so to speak, in the mysterious world of higher mathematics.

Confusion concerning the nature of the infinite arose as soon as man attempted to impose his mental order on the world in which he lived. In the 4th century BC, the Greek skeptic Zeno delighted in confounding his supposedly sophisticated colleagues with paradoxes of motion. A moving object like an arrow cannot actually traverse any distance during one instant (or point) of time. So how then can it ever move? If you add together even infinitely many instants, during each of which the arrow has not covered any ground, how can you get the large distance that is actually determined by the arrow's path? Zeno asked, in short, whether it is possible to add up infinitely many infinitely small things.

This is exactly what is done in the subject of calculus, the formal entry to the field of higher mathematics. In fact, calculus used to be called "infinitesimal calculus", especially in the years shortly after its creation by Isaac Newton and Gottfried Leibniz in the latter part of the 17th century, because one of its central themes is the aggregation of infinitely many infinitely small rectangles to compute areas of regions with curved sides.

Despite the enormous success of calculus, it has been plagued for three centuries by a serious logical flaw: the notion of an infinitesimal (that is, an infinitely small number) contradicts the basic axioms of numbers. Zeno sensed this in posing his paradoxes. Aristotle, following Zeno, argued firmly that reasoning with infinitely large or infinitely small numbers is absurd—because it necessarily leads to paradoxes and contradictions. Leibniz tried to avoid the stigma of employing logically contradictory notions by calling his infinitesimals "useful fictions", but Newton's critic Bishop Berkeley dismissed them as "ghosts of departed quantities" and went on to argue that anyone who could believe in them need not be "squeamish about any point in divinity".

The problem with infinitesimals—first articulated precisely in the latter part of the 19th century—is that they are neither large enough to behave like positive numbers, nor small enough to act as if they are truly zero. One of the fundamental properties of numbers is that if you add together sufficiently many of them, no matter how small they are, you can get enormously large numbers. That is how gross national products measured in trillions of dollars are formed from each person's two-cents-worth of contribution, and how astronomical distances can be measured using simple units such as inches or centimetres.

But by its very nature an infinitesimal (that is, an infinitely small number) cannot have this property. No matter how many infinitesimals you add up, the sum will always remain infinitesimally small. The only infinitely small number allowed in the classical theory of ordinary mathematics is the number zero—and adding up lots of zeroes never gets you anywhere. Yet those who used infinitesimals (in calculus, for instance) required that they be large enough to add up to specific numbers (areas, for instance) while still small enough to behave like zero in other contexts.

In the latter part of the 19th century, mathematicians finally fought their way out of the thicket of contradictions at the base of calculus and proposed an elaborate theory that re-explained all the concepts involved in calculus—indeed, in nearly all parts of mathematics—without resorting to any mention of either infinitely large or infinitely small numbers. In the 20th century mathematics has been governed entirely by the tenets of this "classical" analysis: a firm understanding of the 19th century "infinitesimal-less" theory of calculus is still a basic prerequisite for admission to any graduate school in mathematics.

A radical reinterpretation

As countless former students of calculus will testify, ordinary "classical" analysis is a strong brew. It has been distilled from three centuries of work originating in the towering genius of Isaac Newton. Yet for all its sophistication classical analysis completely fails to explain the intuitive power of infinitesimals. This is what the new "non-standard" analysis achieves: by a radical reinterpretation of the language of mathematics, non-standard analysis restores respectability to reasoning with infinitesimals. It does this by building not only on the foundation of classical analysis, but also on the ethereal yet profound concepts of 20th century logic.

The conduct of logical reasoning from premises conclusions is one of the profound legacies of ancient Greek culture. All scientific and philosophic analysis fits this intellectual template, not least mathematics. Indeed, the archives of mathematics consist largely of various systems

Isaac Newton

of carefully stated axioms, deductions that follow logically from them, and models that conform to the axioms. Nonstandard analysis is, roughly, a new model for old axioms.

Since a mathematical model is simply a mental construct, it must be specified in great detail in order to insure that all persons will have the same thing "in mind" when reasoning with the model. Mathematicians cannot rely on empirical examples as the basis for the objectivity of their science. But neither can they rely solely on the power of the axiomatic method. In 1931 Kurt Gödel showed that no system of axioms can describe completely and unambiguously a sophisticated and complex model: no matter what axioms we write down for, say, elementary arithmetic, there will be arithmetical statements that are true in the intended model but cannot be proved on the basis of the axioms. In other words, the axioms for arithmetic cannot distinguish among various models that differ in subtle ways. No axiom system (except a very simple one) provides a complete description of the model it is intended to describe.

Describing the infinite

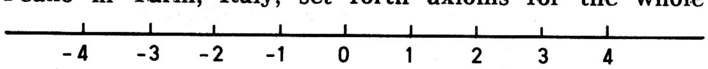
The problem, it turns out, is due to our inability to describe precisely infinite things in a finite language. Virtually every useful mathematical theory involves infinite sets, yet all propositions, axioms and proofs must be of finite length. This limitation on language—that each human utterance, including each logical demonstration of a mathematical theorem, is of finite duration—imposes a severe limitation on the power of axiomatic systems. The finite nature of mathematical language is the basis for Gödel's proof, and, as we shall see, for all of Robinson's work concerning the infinite and the infinitesimal.

The major growth of mathematics has always taken place in its upper branches, where new buds open into blossoms of unexpected power and beauty. But the roots of mathematics grow too, especially those that tap the psychological resource of the basic idea of "number." Number means precisely what mathematicians define it to mean, but their definitions continually evolve to reflect their improved understanding.

Prehistoric man began using numbers as a simple counting device. Greek geometers interpreted number as length, and were puzzled by "incommensurable" numbers that did not share a common measure with the ordinary whole numbers. Mathematicians of the 17th and 18th century wrestled with other exotic numbers and gave them such pejorative labels as negative, imaginary, irrational, radical. But despite inertia and misunderstanding—even some famous 19th century mathematicians had trouble multiplying negative numbers propertly—the meaning of "number" gradually enlarged until it reached a now-classical plateau by the end of the 19th century.

The most important system of numbers in this classical scheme is the system known as the "real numbers." (They are called real to distinguish them from the imaginary numbers that involve the square root of -1.) The real numbers are represented graphically by a ruled straight line extending without bound in both directions (Figure 1). The number 0 is at the centre, positive numbers to the right and negative numbers to the left. The real number line is, simultaneously, the idealisation of time, and of one-dimensional space. It is constantly used as one dimension in a graph, representing either space or time.

In classical analysis the real number line is constructed by extending the intuitively self-evident properties of the whole numbers to an enlarged domain. In 1890 Giuseppe Peano in Turin, Italy, set forth axioms for the whole



The straight line representation of the set of all real numbers

numbers *0, 1, 2, 3, . . .* which thus formed the core of the axioms for the real numbers. In essence, Peano described the whole numbers by the property that each number has an immediate successor, and that each number—except the first one, *0*—has an immediate predecessor. He thought, naively, that his axioms provided a complete and unambiguous description of the infinite set of whole numbers.

But he was wrong. In 1934 the Norwegian logician Thoralf Skolem showed that there must be models other than the one Peano intended that also satisfy his axioms. Skolem created a model that had, in addition to the ordinary numbers, a whole range of infinite numbers, sometimes loosely described as "theological numbers". These are ideal objects that exist on the other side of the ". . .". To do this, he simply postulated one infinite number (suppose it is called ω) and then used Peano's axioms to generate a whole range of predecessors and successors:

$0, 1, 2, 3, \ldots \qquad \ldots \omega\text{-}2, \omega\text{-}1, \omega, \omega+1, \omega+2, \ldots$

Skolem's model for Peano's axioms was the first nonstandard model in mathematics.

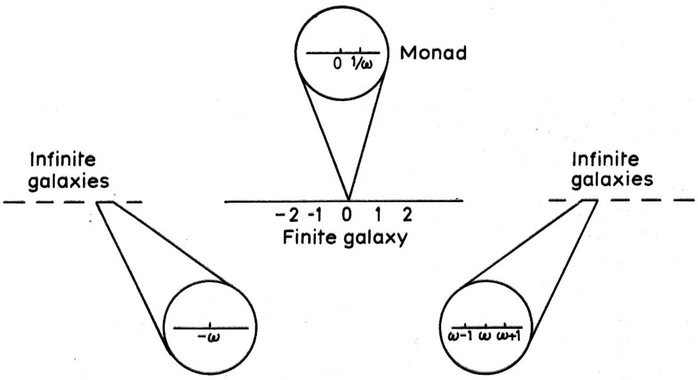
About 20 years ago, Abraham Robinson, while consulting Skolem's original paper, realised "in a flash" that it might be possible to think in terms of infinite numbers relative to the real number line the way Skolem had done for the whole numbers. By examining his idea more carefully he discovered that this route would be the best way to give a coherent and consistent theory of the discredited infinitesimal as well.

What Robinson did, imitating Skolem, was to imagine the existence of one single infinite number, and then let the ordinary axioms of the real numbers draw out the consequences of this idea. This infinite number (let's call it ω again) would have to have, for instance, all sorts of other infinite numbers around it—corresponding to what one would get if one were to add various other (finite) numbers to the infinite one. In addition, the model would have to contain the reciprocal of the given infinite number (that is, $1/\omega$), because all real numbers (except for zero) have reciprocals. This reciprocal of an infinite number is an infinitesimal.

The nonstandard real line is to the standard real line as the Universe is to our Galaxy. Instead of having just one line, the nonstandard model turns out to have infinitely many lines extending both to the right and to the left beyond the "ends" of the standard real line. These new copies of the real line are called "infinite galaxies", and the standard line the "finite galaxy." This is the macroscopic view of the nonstandard real line; there is a microscopic view as well.

If we imagine examining the ordinary real line under a microscope of infinite power, we will, according to classical theory, discover that the line is composed of points. But in Robinson's non-standard model, when the microscope is focused on one of these points, it reveals that the point is not just a simple point, but a tiny copy of the ordinary real line. (This discovery is akin to the discovery that atoms are not really indivisible, but consist of electrons, protons, and neutrons.) This tiny copy of the real line is called a monad (in honour of Leibnitz), and the distances measured within a monad are infinitesimals. The non-standard real line can therefore be visualised geometrically as a complex structure of worlds within worlds, with galaxies comprised of monads spread out in infinite distances (Figure 2).

The actual construction of the non-standard model is more a matter of language than of geometry. Although the geometry helps people perceive the structure—as the ordinary straight line with ruler marks on it helps people perceive the structure of the ordinary real line—the essence of a non-standard model lies in a careful analysis of just what can and cannot be said in ordinary language about mathematical models.



The non-standard model developed by Robinson treats each "point" on the line representing real numbers as a tiny copy of the real line

Robinson began by listing all basic propositions about numbers that can be stated in a simple formal language—such as the one ordinarily used by mathematicians in their daily work. He then added to this collection a whole list of propositions that describe infinitesimals and infinite numbers, and finally used a powerful theorem of mathematical logic to find a model for the enlarged theory. This theorem, known as the compactness theorem, sets forth broad conditions—easily met in this particular instance—under which lists of axioms must necessarily have a model.

Robinson called the standard real line *R*, and the non-standard one **R*. The surprising and most important fact about **R* is that it is a model for the same elementary theory as is *R*: any true thing that can be said in the elementary language about *R* applies also to **R*. This so-called "transfer principle" seems incredible ("double talk" according to one of the physicists who is now applying the theory), since in **R* there are infinite numbers that are not in *R*. But in the non-standard world, words such as "finite" and "infinite" turn out to have meanings different from those they have in ordinary mathematics. For instance, whenever a mathematician tries to describe the finite numbers in **R*—that is, the whole numbers that are in the finite galaxy—the language in which propositions about **R* are expressed forces him to accept as a "finite number" a lot of numbers from the infinite galaxies as well.

The reason for this slippery state of affairs is that the formal language in which mathematical thoughts are expressed—that very same formal language that Robinson began with in his construction—is incapable of capturing every nuance of the models our minds can envisage. Whenever mathematicians begin to deal with infinite sets—and virtually every mathematical model does because the set of numbers itself is infinite—they run up against the finite limitation of natural language: all human utterances, including every statement and proof of mathematical propositions, must be of finite length. And finite descriptions are incapable of providing complete detail for infinite models.

The result is that differing models may be indistinguishable in the vocabulary of formal mathematics. Indeed, every formal statement about a non-standard model corresponds to an identical formal statement about an ordinary model. Thus there is, in the literal sense, nothing new in the non-standard universe: what differences there may appear to be lie entirely in the eyes of the beholder. This unseemly subjectivity, claim critics of non-standard analysis, is inappropriate in mathematics. Since there can be no *formal* difference between the two systems, old and new, the non-standard models appear to some to be inadvisably replacing concepts steeped in centuries of physical intuition with ethereal abstractions rooted only in the mathematically thin soil of modern logic. Stanford mathematician Errett Bishop, reviewing a new calculus book using the non-standard approach, laments that non-standard analysis will only confirm students' "experience of mathematics as an esoteric and meaningless exercise in technique".

While this criticism may be true in the literal sense, it really applies only to one of several areas in which mathematics contributes to our understanding, namely to mathematics as a system of deductions based on axioms. But mathematics is not only a body of results; it is also a language and a method. Whenever the language of mathematics is simplified so as to harmonise its formal concepts with either intuitive or physical insight, scientific discovery is advanced.

Mathematicians who encounter non-standard models for the first time become frustrated because they feel they have lost control over the words they use; words no longer mean precisely what they intend them to mean. But the silver lining in this cloud is enormously valuable, for all the well known facts and intuitive judgements about finite structures apply in non-standard models to certain infinite structures as well. Non-standard analysis thus turns a liability into an asset by using the inability of elementary language to distinguish properly between the finite and the infinite as a bridge for our intuition between those two distinctly different realms.

Precisely three centuries after Isaac Newton introduced the calculus of infinitesimals as a tool for those seeking to comprehend the laws of nature, Abraham Robinson re-created a "non-standard" calculus of infinitesimals as a tool for those seeking to understand the process of mathematical reasoning. Whereas Newtonian analysis eventually reduces the infinite to the finite, Robinson reversed this flow by extending the finite to the infinite. Non-standard analysis may well be the analysis of the future. □

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