NEWS & LETTERS

SOLUTION OF THE FOUR COLOR PROBLEM

The four color conjecture, one of the most popular and appealing unsolved problems of mathematics, was verified this summer by an intricate computer-based analysis carried out by Kenneth Appel and Wolfgang Haken of the University of Illinois. While it may take a year or longer for others to verify every detail of their work--the proof contains several hundred pages of what even the authors term "ridiculous detail" and subsumes over 1000 hours of computer calculation--the general outline of their method and the initial confirmation of their major calculations is accepted by most graph theorists as complete and correct.

The conjecture was first posed in 1853 by Francis Guthrie, a mathematics student at University College, London: can every map in the plane be colored with four colors so that adjacent regions receive different colors? The first "proof" was published in 1879 by A.B. Kempe, but it proved to be incorrect. Appel's and Haken's new proof is simply a very elaborate correction of Kempe's oversight.

Kempe began by showing, correctly, that it suffices to verify the conjecture for "normal" maps in which precisely three regional boundaries meet at each vertex: vertices where more than three boundaries meet can be separated into several trivalent vertices, and the resulting map will be more difficult to color than was the original because more regions are adjacent. He then used Euler's formula $V - E + F = 2$ relating the vertices, boundaries (or edges) and regions (or faces) of a map to show that any normal map must contain regions with fewer than six neighbors.

This isn't too hard. Let e_i denote the number of edges of region i. Then the total number E of edges is precisely $\frac{1}{2}\Sigma e_i$. Moreover, in a normal map, $2E = 3V$, where V denotes the total number of vertices. Hence

$$\Sigma(6-e_i) = 6F - 2E = 6F - 6E + 4E$$
$$= 6F - 6E + 6V = 12.$$

If each e_i were six or greater, the left side would be zero or less. Hence some region must have fewer than six neighbors.

Armed with this information concerning "unavoidable" regions, Kempe tried to show that whenever a region with fewer than six sides appeared in a normal map, the map could be reduced to a smaller one whose coloring was no easier if the reduced map could be four-colored then so could the original map. His proof was simple and correct for regions of 2, 3 or 4 sides, but for pentagonal regions his proof was incomplete: in 1890 P.J. Heawood gave an example of a normal map with 25 regions that contained a pentagon that could not be reduced according to the methods used in Kempe's proof. Heawood's map could, of course, be colored, but not by Kempe's method.

Heawood's analysis of the Kempe proof demonstrated that the problem was far more subtle than had at first been believed. The problem attracted the attention of amateur and professional mathematicians throughout the world. But not even the best minds of the twentieth century could solve it. Laborious modifications of Kempe's reducibility arguments revealed only that maps of size no larger than 40 or so must be four-colorable. These efforts climaxed, in a way, with Martin Gardner's publication of a hoax counterexample in the April 1, 1975 issue of *Scientific American.*

Coloring problems are at present customarily translated into graphs by duality: each region is replaced by a point in it (the capital of the country, to speak geographically) and each boundary by a line joining two points. Then a normal map becomes a triangulated graph (one whose faces are all triangles), and the Euler formula for graphs leads to various Kempe-type reduction arguments.

One particularly convenient way to work with the Euler formula on triangulated graphs is to assign a "charge" to each vertex v of $6 - n(v)$, where $n(v)$ is the number of edges that meet at vertex v. Then the Euler formula implies, as above, that the sum of these charges for any triangulated graph is 12. Since positive charge occurs only on vertices of degree less than 6 (i.e., in regions with fewer than six neighbors), we can be sure--from the positive total--that such vertices are unavoidable in triangulated graphs. Kempe tried unsuccessfully to show that this set of unavoidable configurations was also reducible; where Kempe failed, Haken and Appel succeeded. But to do so they had to replace his one flawed case (pentagonal regions) with 1,936 complex configurations.

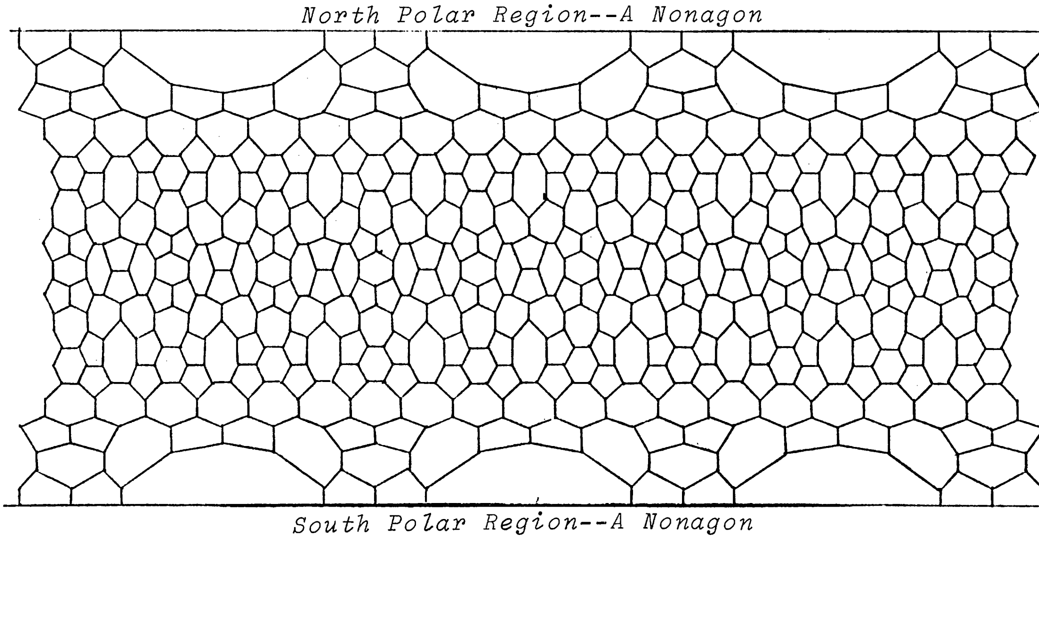
The strategy behind the Appel-Haken proof is to redistribute the Eulerian charge on the graph using a specially designed "discharging algorithm" in such a way as to minimize the number of positive vertices--always subject to the constraint that the total charge must remain unchanged. Then each of the configurations identified by the remaining positive charges must be reduced, just as Kempe had reduced the cases of 2, 3 and 4 sided regions.

The first step in this program--the design of an appropriate discharging algorithm--took about three and one-half years of counterpoint between man and machine. For each vertex with positive charge produced by the draft algorithm, Haken and Appel tried to find a reducible configuration around it (using specially designed computer algorithms for assistance). If no reducible configuration was discovered in reasonable time--perhaps half an hour of computer search--they assumed that none existed and went back to modify the discharging algorithm to avoid such situations.

When they were finally convinced that they had an algorithm that located only reducible configurations, Appel and Haken began the systematic verification of reducibility for all cases of positive charge produced by the algorithm. The resulting catalogue contained 1,936 reducible configurations, each requiring a search of up to 500,000 options to verify reducibility. This last phase of the work took six months, and was completed in June, 1976.

Final checking--part of which was carried out by the researchers' teen-age children--took the entire month of July,

North Polar Region--A Nonagon



South Polar Region--A Nonagon

and the results were communicated to the *Bulletin of the American Mathematical Society* on July 26, 1976: "Every Planar Map Is Four Colorable." This official announcement is scheduled to appear in the September 1976 issue of the *Bulletin*.

The Haken-Appel proof has sent several different shock waves through the mathematical world. The verification of a century-old conjecture that had baffled the twentieth century's best mathematicians is an astounding accomplishment. But a solution based on computerized case analyses involving nearly 2000 cases and 10 billion logical options is the complete antithesis of the idealized "elegant" matheatical proof. (Perhaps this is why Haken's personal presentation of the result to an audience of several hundred mathematicians at the University of Toronto in August was greeted with no more than mildly polite applause.)

The Haken-Appel proof is the first example of a major mathematical problem solved by an essential symbiosis of theoretician and computer. Many mathematicians feel that this result is only the prelude to better, shorter, more conceptual proofs. "We aren't going to go through eternity," vowed one mathematician, "saying 'And the computer said...'."

In defense of their formidable method, Haken and Appel observe that theirs is close to an optimal proof within the Kempe tradition of seeking unavoidable sets of reducible configurations. Every configuration that must be reduced is surrounded by a ring of neighbors that determine its reducibility. The size of this ring has great bearing on the difficulty of establishing reducibility.

Several years ago Edward F. Moore of the University of Wisconsin developed a strategy for disproving the conjecture (if indeed it were false) by creating maps that exclude all known reducible configurations almost as fast as such configurations were discovered. The map on p. 220 (flattened out from the surface of a sphere: each polar region bounded by the top and bottom lines is a nonagon) is one Moore created in 1963 that contains no reducible regions whose ring size is smaller than 12. The Haken-Appel proof requires configurations of ring size no larger than 14.

The Moore graph, therefore, shows that no proof based on an unavoidable set of reducible configurations can be even moderately short, since it must deal with ring sizes at least as large as 12; the Haken-Appel proof, while somewhat longer than necessary, deals with configurations only two sizes larger.

The fact that the Haken-Appel proof appeared between ring sizes 12 and 14 confirms some general probability estimates concerning the likely occurrence of reducible configurations in randomly drawn plane maps: it is quite likely that maps do exist that contain no reducible configurations of ring size smaller than 13, but it is also very likely that every map contains a reducible configuration of ring size not exceeding 14. These estimates establish upper and lower bounds on the length of any Kempe-like proof of the four color problem. Haken and Appel's proof fits right in between these theoretical limits.

Speculators on the market of mathematical problems might be inclined now to support computer attacks on all famous unsolved problems. But the crucial first step in any computer attack is a difficult theoretical maneuver --the reduction from an infinite to a finite number of cases. This is possible in the four color problem because of the intricate geometry of maps: the behavior of graphs of size n very strongly influences the behavior of graphs of size $n + 1$. With luck and insight, it is possible to develop a finite number of cases that cover all infinitely many possible maps.

Such is not likely to be the case with problems in number theory such as Fermat's conjecture, for the behavior of prime numbers appears to be much more loosely knit than is the geometry of graphs. So the finitization of problems in number theory will either be very difficult or perhaps impossible. And no computer assault can work until the finitization theory is complete.

But even if a problem is finite, it may be impractical to implement on even the fastest computers. Had the Haken-Appel attack turned out to require configurations of ring size 15, the time required for computer search of reducibility would have made the present proof totally impossible.

The Haken-Appel result points, therefore, to the existence of a new class of mathematical theorems that are true, but for which no simple proof exists. Exploration of this realm by mathematician-computer teamwork is, according to Haken and Appel, a major challenge for mathematicians in the final quarter of this century. "This work has changed my view of what mathematics is," said Haken. "I hope it will do the same for others."

--Lynn Arthur Steen

LESTER R. FORD AWARDS

Authors of six expository papers appearing in the 1975 issue of the *American Mathematical Monthly* and *Mathematics Magazine* received Lester R. Ford Awards at the August meeting of the Mathematical Association of America at the University of Toronto. Each award is in the amount of $100. The award winning papers are:

M.L. Balinski and H.P. Young, The Quota Method of Apportionment, *Amer. Math. Monthly*, 82 (1975) 701-730.

E.A. Bender and J.R. Goldman, On the Applications of Mobius Inversion in Combinatorial Analysis, *Amer. Math. Monthly*, 82 (1975) 789-803.

Branko Grunbaum, Venn Diagrams and Independent Families of Sets, *Mathematics Magazine*, 48 (1975) 12-23.

J.E. Humphreys, Representations of LS(2,*p*), *Amer. Math. Monthly*, 82 (1975) 21-39.

J.B. Keller and D.W. McLaughlin, The Feynman Integral, *Amer. Math. Monthly*, 82 (1975) 451-465.

J.J. Price, Topics in Orthogonal Functions, *Amer. Math. Monthly*, 82 (1975) 594-609.

STATISTICS LECTURERS

The Visiting Lecturer Program in Statistics, now its fourteenth year, attempts to provide information to students and college faculty about the nature and scope of modern statistics, and to provide advice about careers, graduate study, and college curricula in statistics. Leading teachers and research workers in statistics--from universities, industry and government--have agreed to participate as lecturers. Lecture topics include subjects in experimental and theoretical statistics, as well as in such related areas as probability theory, information theory and stochastic models in the physical, biological and social sciences.

The Visiting Lecturer Program is sponsored by the American Statistical Association, the Biometric Society and the Institute of Mathematical Statistics. Partial support is also provided by International Business Machines Corporation. The program is available to colleges and universities in the United States and Canada. Inquiries should be addressed to H.T. David, Visiting Lecturer Program in Statistics, Department of Statistics, Iowa State University, Ames, Iowa 50011.

SCHOOL MATHEMATICS COMPETITIONS

The New York City Interscholastic Mathematics League currently holds five contests each semester. Although official entrance into the NYCIML is limited to secondary schools in New York City, the league welcomes unofficial entry by schools from outside New York City. The dues are $15 per team per semester.

Any school interested in joining the league on an unofficial basis, or in using its problems as the basis for their own minileague, may do so by sending a check for one full year's dues. Those who wish to receive one copy of each of the contests on a regular basis can do so for a $5 league fee, payable in advance.

Further details can be obtained from Steven R. Conrad, President, NYCIML, 39 Arrow Street, Selden, New York 11784.