Foundations of Mathematics: Unsolvable Problems

Mathematicians have known since 1931 that some exotic mathematical problems must necessarily be unsolvable, but only within the last decade did they begin to discover examples of such problems in many parts of mathematics. Now hundreds of such problems have been proved to be unsolvable. Recently two rather famous problems—one proposed by the German mathematician David Hilbert in 1900 and the other proposed by the Russian mathematician Mikhail Souslin in 1920—have been added to the growing list.

Actually, there are two distinct types of "unsolvability" in mathematics. One kind, illustrated by the 19th-century result that the classical Greek problem of trisecting an angle is unsolvable, is really an instance of "impossibility." The other type, of far greater scientific and philosophic import, is really a judgment of "undecidability": the discovery of non-Euclidean geometry showed, for example, that Euclid's fifth (parallel) postulate could not be decided—that is, proved or refuted—on the basis of the other accepted axioms of plane geometry.

The possibility—indeed, the certainty—that some mathematical problems may actually be undecidable was first discovered by the logician Kurt Gödel, now at the Institute for Advanced Study in Princeton. He showed in 1931 that all axiomatic systems (except very simple ones) must contain assertions that can be neither proved nor refuted by logical deduction from the given axioms. This means that all of the famous unsolved problems of mathematics—the four color problem, Goldbach's conjecture, Fermat's last theorem, and so on—became candidates for the purgatory of perpetual undecidability, and that mathematicians will have to determine whether they are undecidable or merely very hard to solve.

The first major breakthrough in the search for specific undecidable propositions came in 1963. In that year Paul Cohen of Stanford University, extending work begun by Gödel in 1939, established the undecidability of a conjecture due to the 19th-century mathematician Georg Cantor concerning the relative sizes of subsets of the real number line. Cantor was trying to formulate a concept (now called cardinal number) that would permit comparative judgments about the sizes of infinite sets. He conjectured that every subset of the real numbers must have the same size either as the set of all integers or as the much larger set of all real numbers.

Cantor's so-called continuum hypothesis took nearly two-thirds of a century to resolve, and then Cohen found that the resolution was neither a proof of the conjecture nor a counterexample to it. It was, rather, a revolutionary analysis of the limitations of logical reasoning leading to the conclusion that Cantor's conjecture can be neither proved nor disproved on the basis of the accepted axioms of set theory.

Cohen's method of proof, the basis for most undecidability results, is a delicate chain of reasoning in which one very carefully forces into existence a mathematical model with certain predetermined properties. The method has been applied extensively during the past decade to construct mathematical models with all sorts of exotic properties and, in the process, establish the undecidability of a host of mathematical propositions, some obscure and some rather famous. Each undecidability proof requires construction of a model in which the proposition in question is true and of another one in which it is false: the undecidability of the proposition follows from the existence of such models, for no general proof or refutation will be possible if the proposition is, in fact, true in some models while false in others.

A few years ago, Thomas Jech of the State University of New York, Buffalo, and Stanley Tennenbaum of the University of Rochester found—by modifying Cohen's method—a model in which Souslin's conjecture is false. Souslin's conjecture, like Cantor's, had something to do with the size of the real number line. What Souslin proposed was a simple characterization of the real number line—a specific axiomatic description that would logically entail all properties of the real number line.

Souslin knew that the ordinary properties of the real numbers—their arrangement in a linear order without any gaps, for instance—are not adequate to unambiguously characterize them because there are mathematical structures that have all of the ordinary properties but are quite different from the real numbers. One such structure is called the long line because it looks just like the ordinary real line but is much longer: it contains the ordinary real line as a tiny subset. Souslin conjectured that the real line is, in a certain sense, the smallest object that satisfies all the ordinary, arithmetical properties of numbers.

The model of Jech and Tennenbaum did not disprove Souslin's conjecture, for to do that one would have to demonstrate that the conjecture is false in every possible model. Jech and Tennenbaum only found one model in which it was false. But then Ronald B. Jensen of the University of California, Berkeley, using still another variation on Cohen's method, found a specific model in which Souslin's conjecture is true. The existence of these two models with opposite features ensures that Souslin's conjecture is actually undecidable.

Like Euclid's fifth postulate, Cantor's and Souslin's hypotheses are not decidable from the other conventional axioms. When mathematicians say that they are unsolvable, they mean simply that the axiomatic structure of mathematics is not sufficiently powerful to decide whether they are true or false.

But unlike their equivocation over Euclid's fifth postulate, mathematicians' inability to resolve Cantor's and Souslin's conjectures is not due merely to their refusal to write down sufficiently many axioms. Cantor and Souslin were attempting to describe properties of a large infinite set (the real numbers); the undecidability of these properties is a reflection of the hazards of employing a logical leap of faith to extend our knowledge of finite sets to infinite ones.

In contrast, the problem proposed by Hilbert—specifically, the tenth on the list of 23 problems which he set forth in 1900 as challenges for 20th-century mathematics—is unsolvable in the sense that no objects of the sort required by this problem can ever exist, in theory or in practice. Hilbert asked in his tenth problem for an algorithm (a list of instructions for solving a problem) that could decide for any polynomial equation whether or not it had any integer solutions. In 1970 the young Russian mathematician Yuri Matiyasevich of the University of Leningrad proved that no such algorithm can exist.

Matiyasevich's proof is totally unlike Cohen's forcing methods, and the nature of his conclusion is likewise quite different. Matiyasevich succeeded, by means of a complex Diophantine equation (one whose solutions are required to be integers), to reduce Hilbert's tenth problem to a classical argument concerning the nature of algorithmic processes: there is no general method which can be used to determine whether a proposed algorithm will necessarily halt—that is, yield an answer. This result, popularly called the halting problem, depends on reasoning analogous to that used by Gödel in his proof that axiomatic systems must have undecidable propositions.

All these results—Gödel's undecidability theorem, the halting problem, and Matiyasevich's answer to Hilbert's tenth problem—employ a "diagonalization" technique first introduced by Cantor to prove that the set of real numbers was too large to be ennumerated even in a potentially infinite list. Bold variations on this single theme produce a family of related impossibility results: we cannot decide all propositions, we cannot decide whether a computer program will necessarily produce an output, and we cannot determine whether polynomial equations necessarily have integer solutions.

Classical mathematics had its share of impossibility results too: we cannot trisect an angle by Euclidean means, we cannot find a formula to solve exactly all polynomial equations of degree greater than five, we cannot square a circle. The new results are analogous to these old ones, but far more general. Instead of saying that some one problem cannot be solved, they are saying that whole classes of problems cannot be solved. They are, in a very fundamental sense, a statement of certain limits on man's intellectual ability.

Hilbert concluded the address in which he set forth his 23 problems with the affirmation: "We hear in us the perpetual call: There is the problem. You can find it by pure reason, for in mathematics there can be no *ignorabimus*." Centuries before Hilbert, juries often returned the verdict of *ignorabimus* (we will not know) when they found the evidence insufficient for a verdict. When faced with undecidable propositions, the jury of contemporary mathematicians has also begun to render the *ignorabimus* verdict. Undecidability is no longer a curiosity but a central fact of mathematical research.

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