The Metamathematical World of Model Theory

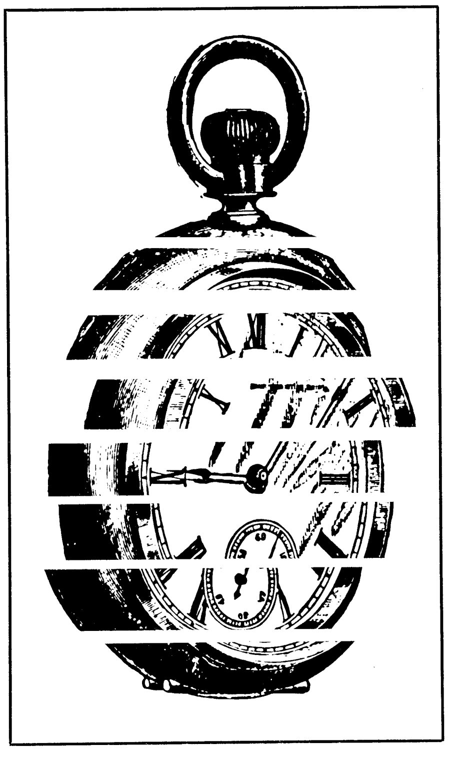
This relatively new theory, ripe for potential research and application, may be a powerful new mathematical tool for analyzing many kinds of scientific problems

by Lynn Arthur Steen

Georg Cantor's theory of infinite sets introduced into early 20th-century mathematics certain devastating paradoxes that posed serious challenges to the logical foundations of mathematics. One consequence of this upheaval was the development of a mathematical theory of mathematics itself—a new discipline called metamathematics, akin to metaphysics but far more rigorous and technical in its methodology. Now, nearly half a century after its creation, aspects of metamathematics are beginning to move beyond the analysis of existing mathematics to the creation of surprising new conceptual models. These models give promise of well representing certain large-scale scientific problems and may even yield a new perspective on the nature of time.

Details of this relatively new theory, known as "model theory," were outlined by H. Jerome Keisler of the University of Wisconsin in four Colloquium Lectures at the annual meeting of the American Mathematical Society in Washington in January. According to society President Lipman Bers of Columbia University, an invitation to deliver the Colloquium Lectures is one of the highest honors in mathematics. Rarely has the subject of these lectures been so rich in history and so ripe for potential research and application.

Model theory has developed rapidly during the past two decades as a tool for classifying and constructing models that arise in diverse parts of mathematics. The subject of its study is mathematics itself, and the tools are highly exotic methods of mathematical logic, particularly various techniques for constructing, enlarging or modifying mathematical models to serve new requirements. As 19th-century models for non-Euclidean geometry and non-commutative arithmetic later proved useful (indeed, indispensable) in relativity theory and in atomic physics, so advocates of model theory foresee model theory providing valuable non-



standard models for many current scientific problems.

The models provided by model theory come, of course, from within mathematics itself. Mathematics is comprised of a *language* of symbols in which sentences may be formed that describe properties of certain abstract objects called models. For instance, by using symbols such as +, =, 0, 1, 2, we may express sentences (e.g., "1 + 1 = 2") about a particular model. What a sentence says about a particular model may be true, or it may be false, or it may be meaningless. A model is said to satisfy a particular sentence if the sentence is true in that particular model. For example, because "1 + 1 = 2" is a true statement about integers, we say that the integers form a model which satisfies the sentence "1 + 1 = 2." Of course other models, the model of ordinary real numbers, for example, satisfy this sentence as well, while still others, such as binary arithmetic, do not.

Axioms, the logical bedrock of mathematics, are nothing but particular sentences in a particular language. A central question in all mathematics, one which assumes special urgency in Cantor's paradoxical theory of infinite sets, is whether in fact there exist models that satisfy particular collections of axioms. Collections of axioms that are logically contradictory or that entail logical falsehoods surely cannot be satisfied by any model, for such a model would have to contain an internal contradiction. Collections of axioms with this failing are called inconsistent and are purged from mathematics whenever they are discovered. Mathematics deals only with consistent collections of axioms, in two essentially different but closely related ways: a syntactical process in which certain sentences (called theorems) are derived from consistent collections of axioms, and a semantical process in which models for consistent collections of axioms are constructed, analyzed and utilized.

The first major result of model theory—a result which provided the primary tool for 40 years of subsequent research—is the so-called Compactness Theorem first proved by the German logician Kurt Gödel in 1930. (Gödel is now a fellow of the Institute for Advanced Study in Princeton.) This theorem establishes certain very general grounds under which the existence of a model can be guaranteed even without knowing what it looks like or how to construct it. Specifically, it says that if every finite subcollection of an infinite collection of axioms has a model, then so must the whole infinite collection.

The power of this theorem lies in its ability to leap logically from evidence based on finite collections to a conclusion that holds for an infinite collection: It is a mathematical bridge

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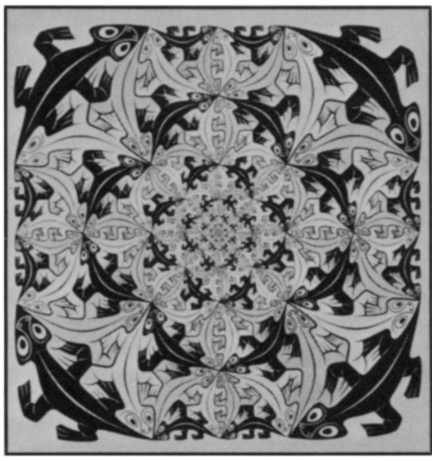
into the transfinite. Mathematicians employ infinite collections of axioms quite frequently, even for such simple things as a rigorous treatment of arithmetic. The Compactness Theorem provides a powerful means for inferring the existence of a model for such an infinite collection of axioms. It is especially useful in enlarging existing models to meet new conditions: The presence of the existing model makes it possible to verify the hypothesis of the Compactness Theorem, thus ensuring the truth of its conclusion.

A second major event in the classical development of model theory occurred in 1935 when the Norwegian logician Thoralf Skolem showed that the theory of ordinary arithmetic must have models other than the intended one—in particular, models containing infinitely large numbers. This idea lay fallow, however, until about 10 years ago when the late Abraham Robinson of Yale University refined it to create a new model for calculus—called "nonstandard" analysis—which incorporated both infinitely large and infinitely small (infinitesimal) numbers.

Isaac Newton and Gottfried Leibniz, the co-inventors of calculus, both employed infinitesimals in their work. But subsequent generations of mathematicians, despite strenuous effort, were unable to find a logically coherent explanation for the behavior of infinitesimals. Nineteenth-century mathematicians, especially Augustin Cauchy and Karl Weierstrass, developed many different arguments designed to show that the entire concept of an infinitesimal was absurd, and they succeeded in banishing it from mathematics as a bit of clever but misguided intuition. Robinson's nonstandard model, however, has reaffirmed the early heroes. It shows, in fact, that Newton's and Leibniz's intuition was not only quite profound but also, in the proper context, capable of rigorous verification.



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"Newton's and Leibniz's intuition about infinitesimals was not only quite profound but also, in the proper context, capable of rigorous verification." M. C. Escher's "Smaller and Smaller" expresses the mystery concerning what happens when things, or numbers, become infinitely small.

Many mathematicians feel that Robinson's solution of the problem of infinitesimal numbers is as significant and revolutionary as, for instance, the 19th-century discovery of non-Euclidean geometries by Bolyai, Gauss, Lobachevski and Riemann. Moreover, his method—an application of basic principles of model theory—may be even more important than his result. Instead of studying details of existing models, he studied as a distinct mathematical entity the language which one uses to describe these models. This is the essence of metamathematics—to study the language of mathematics as distinct from the objects of mathematics, like studying linguistics as opposed to writing a novel. Model theory provides the translation from the language in which mathematics is expressed to the models of which it is comprised.

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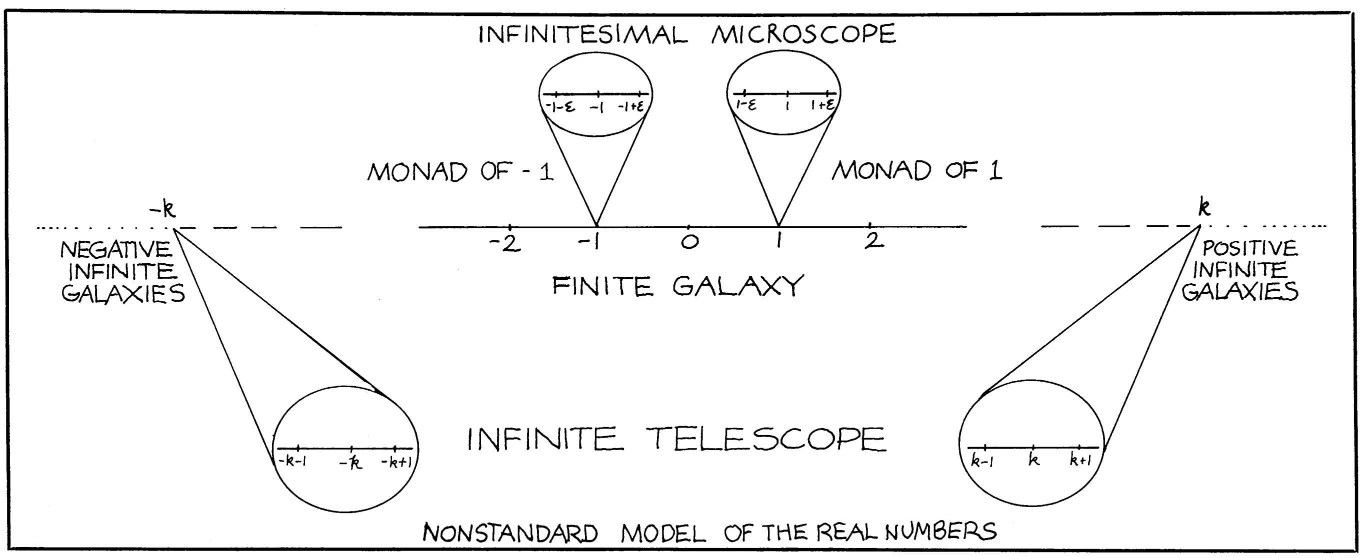
Part of the reason that Robinson's work had greater impact than Skolem's was that he had available more compelling techniques. In 1955 the Polish mathematician Jerzy Los extended one of Skolem's ideas to actually construct (in an abstract sort of way) models of the type whose existence is guaranteed by the Compactness Theorem. Los's model is called an ultraproduct. It is formed by complex algebraic operations (similar to simple products but of infinite extent; hence the name ultraproduct) performed on the elements of more basic models.

The introduction of ultraproducts as a relatively concrete realization of the heretofore totally existential models guaranteed by the Compactness Theorem generated a major resurgence of



On a macroscopic level, the nonstandard universe of real numbers contains, in addition to the ordinary real number line, many "galaxies" just like it which extend on both sides of it. These galaxies appear, when viewed through an "infinite telescope," to be exact replicas of the finite galaxy which is the ordinary real line. On a microscopic level, examination of any point in a galaxy through an "infinitesimal microscope" reveals that it is not just a point, but an infinitely small copy of the entire real line called, after Leibniz, a monad.

interest in model theory. Robinson used ultraproduct constructions to give concrete form to his nonstandard models of calculus: He could actually represent infinitely small numbers by simple sequences of ordinary fractions.

It is this possibility which has led Keisler to develop an entire intuitive introductory calculus course based on Robinson's nonstandard model of anlysis. In this course the routine pattern of reasoning, quite similar to the original intuition of Newton and Leibniz, is overlaid on a tightly woven logical fabric of ultraproducts and applications of the Compactness Theorem.

Model theory offers a powerful mathematical tool for rigorously synthesizing common theoretical conflicts between discrete and continuous interpretations of scientific phenomena. This conflict occurs, for instance, in our conceptualization of time. In the continuous model, with time represented by a straight line, no moment of time is followed by a next moment: Between each two moments there is always an interval of other moments of time. In the discrete model, represented by the counting of years or days or seconds, each moment is separated from the next moment by an indivisible unit of some fixed length.

In the nonstandard model of time, each moment is followed by a next moment, yet two successive moments are infinitely close together. In this model, it is possible to conceptualize a physical or social process as taking place one step at a time, each step infinitely close to the next. Of course, scientists have often thought this way intuitively; the importance of this new model is that, for the first time, such intuitive reasoning can be made logically defensible.

It has practical uses. For example, in his lectures Keisler used this nonstandard conceptualization of time to construct a model of an exchange economy in which prices would be determined by the aggregate effect of a large number of traders acting in a succession of infinitesimally spaced moments. By means of this model he was able to set forth explicit upper and lower bounds on the momentary price fluctuations to make market stability possible. If prices change too slowly, they will not converge to equilibrium in a finite amount of time, while if they change too rapidly, they will never have a chance to become stabilized.

The distinction between the language in which mathematics is expressed and the models about which this language speaks leads to a surprisingly subtle device for utilizing nonstandard models in interpreting scientific phenomena. For example, every nonstandard model of the real numbers contains certain sets, called *-finite (pronounced "star-finite") sets, which appear to be finite when viewed from within the nonstandard model, yet are easily recognized as infinite when viewed from outside the model. The

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reason for this ambiguity is that certain facts about a model (for example, that a particular set is infinite) may be recognized as true by an informed intuition, yet be incapable of expression or proof within the grammar of the mathematical language which describes the model. Semantically, a *-finite set is infinite, but syntactically, it is finite. The formal language of mathematics is simply not sufficiently rich to express all the meaning bound up in its models.

The part-finite, part-infinite character of *-finite sets makes them particularly suitable for use as mathematical representations of very large finite sets such as the voters in an election, traders in an economy, or molecules in a fluid. Robinson and Yale economist Donald Brown have employed *-finite sets to analyze such classical economic problems as Edgeworth's conjecture that as the number of traders in an exchange economy increases, the core of the economy approaches competitive equilibrium. Earlier approaches to this type of problem relied either on a hypothetical sequence of economies growing without bound, or on a model of an economy with an infinite number of traders.

Keisler's most recent work concerns the development of nonstandard probability theory. Traditional elementary probability (used to analyze the flip of a coin or the throw of a die) employs a finite model; more sophisticated and realistic problems require infinite models. Unfortunately, many of the basic methods of analysis that work well in the finite models (e.g., counting, adding) become technically meaningless in the infinite models, even though they often retain considerable intuitive value.

Keisler showed in his lectures that it is possible to substitute for the traditional infinite probabilistic model a nonstandard one based on *-finite sets in which counting arguments may be employed on sets which are, in reality, infinite. Although this investigation is still young, some surprising results have already emerged. One of the more interesting is the discovery of a strong logical connection between two apparently unrelated classical theorems —the law of large numbers from probability theory and the downward Lowenheim-Skolem theorem of mathematical logic.

The law of large numbers says, roughly, that if you repeat an experiment sufficiently often, the average result of your experimental data will almost certainly approach the theoretically expected value. The downward Lowenheim-Skolem theorem is a powerful tool of mathematical logic that ensures the existence of submodels of certain sizes in any given model. Keisler used a * -finite probability model to establish a close connection between these two dissimilar theorems.

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Current research efforts in model theory are focused on both mathematical and scientific problems. One hope of those working in this field is that it may, by construction of appropriate models, resolve some long-standing unsolved problems of arithmetic, for example, the twin prime conjecture which claims that the number of pairs of prime numbers of the form p, $p + 2$ (e.g., 17, 19) is infinite. In applied areas, recent progress has been reported in using nonstandard models in fluid dynamics, quantum field theory and theoretical economics.

It is always difficult to predict the ultimate impact of a new theory. So it is with model theory. The fact that model-theoretical tools are more likely to produce new proofs or new insights than new results—indeed, it is a basic feature of model theory that all formal statements that are true about one model will also be true about enlargements of it, and conversely—leads many mathematicians to question the significance of its contribution to the corpus of mathematical theory. Nevertheless, model theory does produce new models, and thus contributes greatly to the arsenal of conceptual structures from which an applied mathematician must choose his weapon when confronting a complex natural phenomenon. □

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