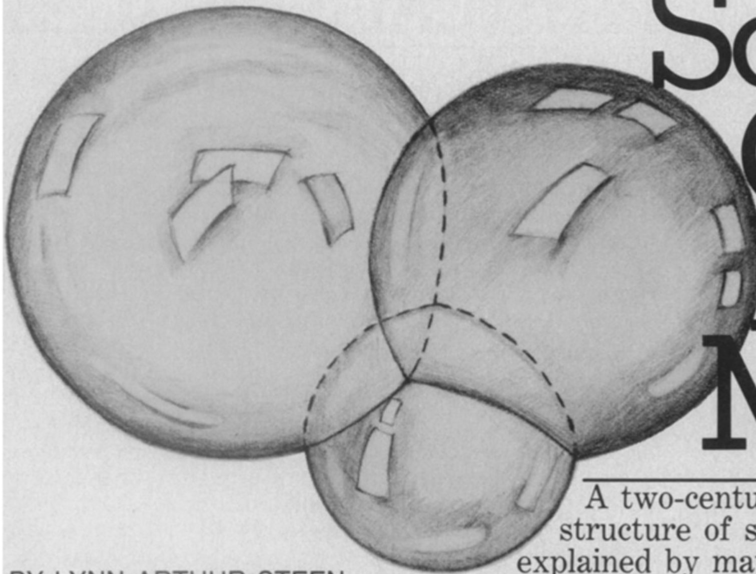


Solving the Great Bubble Mystery

A two-century-old problem—the geometric structure of soap bubble clusters—has now been explained by mathematicians. The solution has applications in many fields.

BY LYNN ARTHUR STEEN

For over two centuries mathematicians and physicists have struggled in vain to explain the geometric structure of soap bubble clusters. Their interest is not just playful curiosity, for the surface tension forces that determine the shape of soap bubbles also affect such diverse natural processes as capillarity, cell growth and annealing of metals.

The search for understanding led investigators to create and explore whole new areas of mathematics, but it never produced a coherent theory that really did explain what makes soap bubbles behave as they do. But now new insights from the mathematical speciality of "geometric measure theory" have made possible a complete solution to this outstanding problem.

The basic difficulty is that no one knew how properly to define the concept of a surface. Simple smooth surfaces posed no problem: The functions studied in ordinary calculus are quite capable of offering complete descriptions of such things as planes (the idealized surface of a table), spheres (the idealized surface of a ball) and similar surfaces with more irregular undulations. But functions offer at most very awkward and inadequate descriptions of more complex surfaces—common among soap films—that have what mathematicians call singularities, that is, edges and vertices caused by self-intersection or branching of the surface.

Surface tension phenomena are governed by a physical principle requiring minimization of surface energy. Since surface energy is proportional to surface area, mathematicians, in addition to figuring out how best to describe surfaces, must also determine appropriate means of measuring their areas. Once this is done, they must then seek a method for finding those surfaces that have least area, thereby minimizing surface energy requirements.

Traditionally, the task of defining a surface has been in the province of geometry, while the study of ways to measure area is part of measure theory—an abstract version of ordinary calculus. The study of the required minimization techniques is part of the calculus of variations—a subject with a rich history going way back to the 18th century. The solution to the soap bubble problem required a new synthesis of geometry, measure theory and the calculus of variations, one which has been achieved only within recent years.

Although several first-class mathematicians such as Newton, Lagrange, Gauss and Poisson worked on theories of surface energy, the soap bubble problem is widely known as Plateau's problem because the first major experimental study of soap film was done by the Belgian physicist J.A.F. Plateau in 1873. Plateau discovered that when soap bubbles meet, they can do so only at angles of 120 degrees; moreover, the singular set of the soap bubble surface (the place where various surfaces intersect) consists of smooth curves along which three sheets of the soap film come together, and points at which four curves meet bringing together six sheets of the surface.

Only by assuming in advance certain special conditions on the possible shape of the soap bubbles were mathematicians able to explain Plateau's observation. Perhaps the most famous such result was that of Jesse Douglas in 1931: His 60-page "solution" to the problem of Plateau—based on the assumption that the soap surfaces are simple enough to be described by functions—was so impressive that in 1936 he was awarded the first Fields medal for this achievement. (The Fields medals, awarded once every four years by the International Congress of Mathematicians, are the mathematical equivalent of the Nobel prize.)

Unfortunately, soap films do not always conform to the assumptions of smoothness and symmetry Douglas needed for his proof. To show theoretically that soap film must fit Plateau's observed structure, it is neither fair nor logical to rule out in advance those subtle cases that make the analysis difficult. A general proof of Plateau's observation would require a full analysis of all possible surfaces, without any extraneous, limiting hypotheses. This general proof was achieved very recently by Jean Taylor of Rutgers University.

Her work is based on results of Frederick J. Almgren Jr. of Princeton University who showed (first in 1967, then in greater generality this year) how it is possible to derive appropriate minimal surfaces from a theory in which surfaces are not limited to those definable by the functions of ordinary calculus. Almgren's methods—reported by him at last month's meeting of the Mathematical Association of America in Kalamazoo, Mich.—involve a subtle redefinition of surface that is tailor-made to fit the subsequent task of measuring its area.

"Measure," in common use, is a verb denoting the act of determining an object's size. Mathematicians use it also as a noun somewhat in the sense of "a one cup measure." A measure, in the mathematician's jargon, is a certain type of mathematical object that measures other objects such as sets or surfaces. A measure is like a specialized computer which, given a set or a surface as input, will promptly print out its length, area or volume—whichever is appropriate.

The key idea that makes the Taylor-Almgren results possible is that surfaces are not merely objects being measured, but are actually measures themselves. This paradoxical but immensely fruitful observation, first formulated about 1960 by Herbert Federer and Wendell H.

Fleming, of Brown University, is revolutionizing the techniques by which mathematicians study properties of surfaces. For when surfaces are considered as measures, it is possible to find ones with properties that just were not possible under the limitations of the older theories.

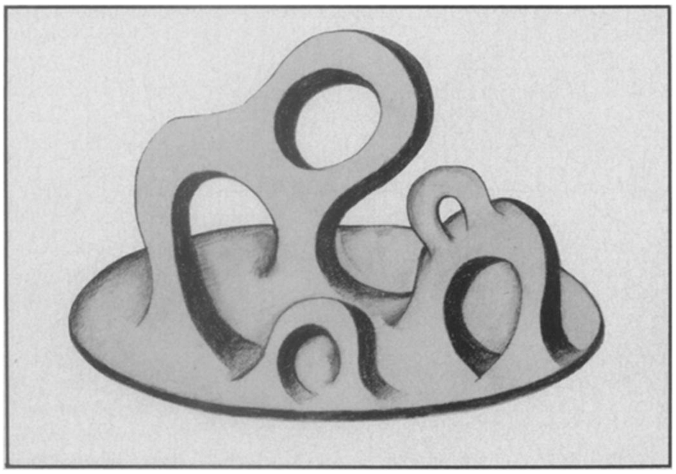
In saying that a surface is really a measure (in disguise), mathematicians are saying only that a surface is an object capable of measuring other things. To see how it can do this, consider a hollow gold sphere sitting inside an otherwise empty box. We want to measure, not the size, but the value of various chunks of the box. To do this, we need only figure out how much area on the sphere is contained in the chunk we are examining, for since the air in the chunk isn't worth anything, all the value of the chunk is concentrated on that part of the gold surface it contains. In this way, the spherical surface provides a means of measuring (the value of) various subsets (chunks) of the box. So it is no misnomer to call it a measure.

Fleming pattern of finding a sequence of measures that converge to the least possible measure. Then they recover the surface from this minimum measure by a very simple device: To locate the surface, sample small chunks of space with the newly found minimal measure. Whenever the measure reports an answer of zero, you know there is no part of the surface in that chunk of space; whenever it is non-zero, you learn where part of the surface is. The minimum measure serves as a type of geiger counter to detect the presence of the surface.

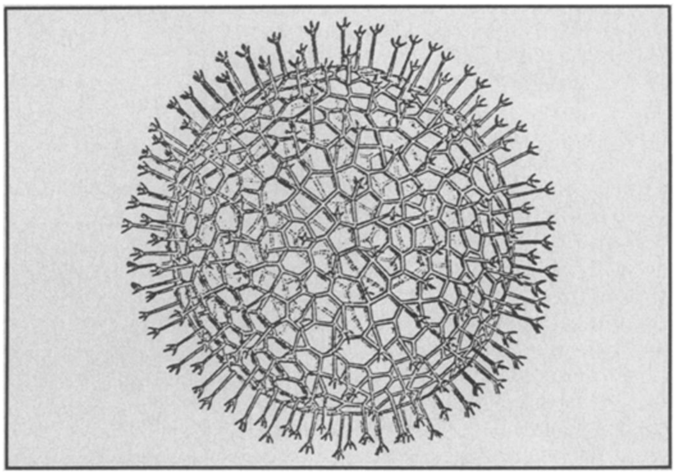
The final step—the one just recently attained—is to show that the surface recovered from this minimal measure is sufficiently smooth and well-behaved to represent actual soap film surfaces. Mathematicians call this part of the problem the regularity proof, and the former part the existence proof. Quite a number of mathematicians have worked on aspects of the regularity proof, including 1974 Fields medalist Enrico Bombieri of Pisa.

structures usually do not mesh properly, so boundary surfaces form—producing an effect somewhat like a froth of soap bubbles. During further slow cooling, these boundaries migrate in a fashion that optimizes the rate at which the crystal surface energy is diminished. The crystal configuration thus changes in a very complex fashion, with some grains vanishing, some merging with others. Almgren's student Kenneth Brakke, in a Princeton thesis submitted just last month, succeeded in using the surface-as-measure theory to analyze this annealing process and to correct some major mistakes in the existing literature on the subject.

Thirty years ago the French mathematician Laurent Schwartz revolutionized the theory of differential equations by introducing "generalized functions" that provide solutions for differential equations that just didn't exist under the constraints of classical analysis. The general structure of Schwartz's theory of generalized functions is very closely analogous to the



Example of complex, self-intersecting surfaces that forced mathematicians to revise their earlier definitions of surface.



Skeletons of Radiolaria, a microscopic form of sea life surrounded by a froth of cells, resemble soap bubble patterns.

The importance of using measure as the definition of a surface is that it allows more general, more wild, more interesting surfaces than do any of the former theories. (In particular, it allows all possible types of soap films.) But by admitting into discourse all sorts of pathological surfaces, it runs the risk of producing existence results that may be physically unrealizable. After all, one of the observed properties of soap films is that the sheets that comprise the bubble complex are very smooth. Almgren's principal discovery is that the results obtained by the surface-as-measure theory are, nearly everywhere, nice smooth surfaces just like those treated in the classical theory. But unlike Douglas, Almgren does not have to assume that his results will be smooth: he proves that they must be.

The techniques of proof are, in outline, rather simple, but the details are horrendous. To find the shape of a minimal surface such as a soap bubble cluster, Almgren and Taylor follow the Federer-

This new approach to problems of minimizing surface energy has already borne surprising fruit in various applications to surface phenomena. Crystal growth, for instance, is determined by minimization of a weighted average over its surface of the energy required to hold the crystal together. Accordingly, crystal growth is modeled well by the surface-as-measure theory. In fact, Almgren has used it to show that in certain situations (example: a sodium chloride cube with one edge cut off) the crystal surface cannot be determined by the surface energy because the resulting equations have no solution. In this case—one that had been studied experimentally long before the theory arose to explain the phenomenon—second-order effects take over and make possible certain unusual and useful results.

Another phenomenon that is covered by this same theory is the annealing of metals. As the metal cools slowly near its freezing point, crystal "grains" begin to grow. When the grains meet, their crystal

"generalized surface" theory of surface-as-measure. In fact, the way Schwartz generalized the classical notion of function was to establish a theory in which functions are viewed as measures.

In the thirty years since its creation, the Schwartz theory has been simplified and added to the basic tool kit of the working applied mathematician. Especially since the new theory of surfaces is so like the theory of generalized functions, those working in the field have every reason to believe that it will quickly be translated from an esoteric new theory to a routine yet powerful method of analyzing and even computing results in the theory of surfaces. □

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