



## Mathematics Achievement Partnership (MAP)

### K–8 Mathematics Expectations

December 2004 Draft

### Commentary

*This December 2004 draft of the MAP mathematics expectations is a substantial revision of the May 2004 draft based on invited reviews received from several individuals and organizations. These notes focus primarily on issues raised by reviewers and provide rationale for my editing decisions.*

*MAP's K–8 Mathematics Expectations is a very complex document that is now in its fourth or fifth incarnation. Notwithstanding months of scrutiny, many errors and unclear explanations undoubtedly remain; almost certainly, through my editing, I have introduced some new difficulties as well. However, the goal of this draft is not perfection of detail, but to provide a version that responds to the most important concerns raised by reviewers. I hope I have succeeded in that objective.*

—Lynn Arthur Steen  
Professor of Mathematics  
St. Olaf College  
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### Organization

**Structure.** The May 2004 draft of MAP's *K–8 Mathematics Expectations* was arranged in nine chapters, one for each grade, K–8. Each chapter opened with four to five *Sample Problem Expectations* to illustrate what students should be able to do at the end of the year. This section was followed by a brief statement of *Major Goals* for the year, accompanied by *Expected Preparation*. The next page provided, in three parallel columns, a *Summary of Expectations* for the previous grade, the current grade and the subsequent grade, each classified into four strands of number and operation, measurement and data, geometry, and algebra. (This feature was especially popular with reviewers.) The summary was followed by the *Expectations* themselves, interspersed with illustrative problems. The final section was a *Problem Book* containing many additional problems, some relating to particular strands, others cutting across several strands.

This December 2004 draft is very different. It contains only the *Expectations*, without any of the introductory summaries or problems. (These other features will be added back into the document before it is published.) Moreover, to focus more readily on issues of mathematical development and consistency, and to provide a fresh perspective for reviewers, *this draft is organized by strand rather than grade*. This way, readers can trace the development of concepts from grade to grade within each of the four strands. Later, the same pages can be rearranged into a grade-level presentation.

**Mathematics Expectations.** This December 2004 draft is organized on four levels of expectations signaled by increasing left indents matched to decreasing font size. The first three levels, with numbers and bullets, contain student expectations stated in different degrees of detail. Items on these three levels contain statements of what students are expected to understand, know and be able to do. The fourth level, with the largest indent and smallest font size, is addressed to educators and devoted to comments that provide a context for the expectations. These are prefaced by the words *Example* or *Note* in bold italic.

*Examples* contain very brief illustrations to clarify the language of an expectation. They are not intended as examples of problems for students — those will be contained in the *Problem Book* associated with each grade — but merely as specific manifestations of general statements that appear in the expectations. *Notes* often contain proofs of statements in the expectations or describe links among different expectations; some contain cautions or suggestions for teachers. Since *Notes* are written for teachers, they often use vocabulary and concepts beyond the current grade level of student expectations.

Expectations given in the first three levels are organized in a hierarchy of decreasing generality and increasing specificity. The top-most level is typically broad (“Understand and use negative numbers”) with various specifics enumerated below as numbered subitems or bullets (“Understand why  $-(-a) = a$ ”). In most cases, subitems and bullets represent extensions or particulars that have some special importance, not a systematic elaboration of the general expectation under which they are situated.

**Chunking.** A distinctive feature of the MAP’s *K-8 Mathematics Expectations* is that topics are addressed in significant chunks rather than spread uniformly across all grades. Although this approach has certain advantages, it created some alarm among reviewers who chaff at the absence of topics they expect to find in particular grades, especially those that provide a scaffold for future topics.

The issue of focused versus distributed topics is not so much about expectations as about curriculum. If, for example, a school district believes that students need to work on proportional relationships in grade 6 to be ready for linear equations in grade 7, it can certainly arrange its curriculum to provide that early preparation — even though the MAP expectation on proportional relationships does not appear until grade 7. Another example concerns the use of literal letters ( $x$ ,  $y$ ) for unknowns.

Some reviewers suggested that they should appear much earlier than they do and be used in every subsequent grade. This December 2004 draft retains the delayed appearance until they are part of a more substantial introduction to algebra. This way, the expectation can be met by teachers who prefer either early or late introduction.

### Generalities

**Progression.** The goal of the MAP *K–8 Mathematics Expectations* is for students to progress from year to year in the several dimensions that constitute mathematical proficiency. These include the four main strands of mathematics used to organize these expectations (number, data, geometry, and algebra) as well as important cross-cutting goals such as reasoning, communicating, modeling and solving problems.

This draft primarily expresses mileposts without elaborating on how they are to be reached. Only in a few cases (e.g., fractions, similarity) does the draft illustrate details of expected progression. Several reviewers mentioned the apparent arbitrariness of where progression was stressed and where it seemed to be ignored. Some argued that the draft does not consistently stress the orderly progress necessary for children to acquire both skills and understanding.

Unfortunately, there is not room to convey details of progression in all areas in an overview document of this nature. State standards and district curriculum frameworks must fill in the details. The fact that progression is elaborated for some topics and not for others is not intended as a signal that progress is important in only a few topics. It is, however, intended as an illustration of how progress in every area can be properly anchored in mathematical thinking.

**Scope and Pacing.** The original goal of the exercise that produced MAP's *K–8 Mathematics Expectations* was to outline annual targets in K–8 education that are necessary to bring students to the performance level suggested in MAP's *Foundations for Success*, the 2001 publication that described what students should be able to know and do at the end of grade 8. In preparing this revision, I benchmarked these *K–8 Mathematics Expectations* against *Foundations for Success* to ensure that most expectations outlined in *Foundations* were at least approximated, if not fully covered, in this draft. Many minor items are different — some added, some omitted — but with only one exception the match generally is rather good. (That exception is “completing the square,” a complex procedure with quadratic functions, which we had previously agreed to drop.)

By approximating the goals set forth in *Foundations for Success*, I knowingly ignored the advice of those reviewers who felt the goals were too ambitious. In most other areas, I sought diligently to take their advice. But here I resisted, for several reasons. First, the rapid pacing of the MAP project has been a well-known goal from the very beginning, so to abandon it would be, in effect, to abandon a fundamental rationale for the entire project. Second, there was some disagreement among

reviewers about the ages at which children can learn different topics, with an experienced minority of both teachers and mathematicians arguing that some MAP pacing was in fact too slow. Third, most reviewers who argued that MAP included too much too fast based their analysis on experience with curricula that are packed with ancillary topics not included in MAP, so it is plausible to imagine that a good teacher working with a more focused curriculum could indeed cover more ground. Finally, the fundamental benefit of MAP is not about rushing to meet all of its expectations by grade 8 but about ensuring that all students see and learn mathematics from the coherent mathematical perspective provided by MAP's *Expectations*. It would be counterproductive to omit topics that are crucial to mathematical understanding just because some students — perhaps even many — will require more time to master them.

**Means versus Ends.** To the extent practicable, these expectations are intended to set goals for student learning and performance at the end of each grade. They are not designed to enumerate all of the steps required for students to achieve these year-end objectives. This singular focus on ends rather than means caused distress among many reviewers who saw the previous draft as incomplete in its coverage of necessary instructional topics.

For example, several reviewers expressed concern that factoring quadratics (in order to solve quadratic equations) is an expectation, yet factoring as a concept or method is not previously introduced. As a skill, factoring is certainly as difficult as many others that are given considerable emphasis, and any curriculum that leads students to success in solving quadratic equations by factoring would need to devote considerable time to mastering the art of factoring. For MAP, however, factoring is only a tool used in solving quadratic equations. It is not — at least not in grade 8 — a separate expectation that would focus on factoring as a general concept and skill.

Factoring is one of many examples of an important difference between a set of expectations and a curriculum framework. *Not everything that may need to be taught and learned to meet a MAP expectation is itself an expectation.* Many topics in school mathematics are means to an end, not ends in themselves. Such topics may be missing from the MAP expectations, but nonetheless must be present in any curriculum that hopes to meet the MAP expectations.

**Problem Solving.** A number of reviewers were frustrated with the May 2004 draft's approach to problem solving. Sometimes problem solving was listed explicitly as part of (or in addition to) an expectation, other times not; sometimes problem solving was restricted to "word problems," other times not; sometimes mathematical problems were included under the umbrella of problem solving, other times problem solving appeared to imply only contextual or word problems; etc.

There is really no dispute about solving problems: *Students should be expected to solve problems of all types using all of the mathematics they study.* The confusion is solely about rhetoric: Should expectations repeat the admonition about solving problems over and over again with every expectation ("... and use it to solve problems"), or just include occasional special sections devoted to solving problems,

or perhaps just highlight problem solving in a preface that overlays the entire set of expectations? Constant repetition dulls the impact of the expectation, while a note about problem solving isolated in a preface that is disconnected from each grade risks being ignored. Generally, I sought the middle ground: Whenever reasonable, I consolidated expectations about problem solving into occasional special sections, but with the understanding that this expectation should apply universally.

**Problem Complexity.** Problem solving is a skill that cuts across strands of number, data, geometry and algebra. As these strands grow in sophistication, the challenge of solving problems increases in cognitive complexity — from straightforward single-step problems to intricate multistep problems in which problem formulation is as much a challenge as problem solution. A general guideline mentioned occasionally in these expectations is worth repeating here: At each grade level, *students should be expected to solve problems that advance in either mathematical content or problem complexity, but not both at once*. In other words, problems requiring new strategies should use calculations that had been mastered earlier, and problems requiring new calculations should use strategies that had been mastered earlier. After each step has been taken separately, then they can be done together — generally in the next grade.

At first glance, this observation appears to be purely pedagogical, something that the MAP *K–8 Expectations* claim to avoid. The goal is to solve complex problems using complex mathematics, and it should not matter how this goal is achieved. It does matter, however, if readers assume that students need to meet all grade-level expectations *at the same time* and incorrectly infer from that assumption that the MAP expectations are impossibly demanding. *Integration of separate expectations into coherent mathematical proficiency — what some have called “profound understanding” — almost always lags individual skills by a year or more.*

## Language

**Verbs.** In earlier drafts of the MAP *K–8 Mathematics Expectations*, the meaning of and distinctions among different verbs created a great deal of confusion. Not only were reviewers critical of seemingly random differences in the nature of verbs used in different expectations (“understand” or “know” or “explain” or “be able to”), but even MAP advisers interpreted these action words in very different ways.

A generic example can be seen in the following list of possible ways to express a single expectation:

- Understand that the product of two negative numbers is positive.
- Understand why the product of two negative numbers is positive.
- Know that the product of two negative numbers is positive.
- Know how to multiply two negative numbers.
- Use the fact that the product of two negative numbers is positive to solve problems.
- Be able to explain why the product of two negative numbers is positive.

Often two or more verbs are combined in a single expectation, as in “understand and use” or “know and be able to.”

Having failed in several previous attempts to clarify the ambiguity surrounding these verbs, in this draft I offer a different approach: simply state the mathematics (“the product of two negative numbers is positive”) without attempting to dictate the desired state of students’ minds (understanding, knowledge, memory, skill) regarding this fact. I adopt this strategy whenever it seems convenient and practical, but not universally. There are still plenty of ambiguous verbs left in these expectations — just fewer than in earlier drafts.

**Adverbs.** Earlier drafts of these expectations were full of advice about how the various verbs were to be carried out, which drew criticism from some reviewers. For example, expectations to “divide efficiently” and “find quickly” made some reviewers imagine a regimen of memorization, while exhortations to “use appropriately” or “understand fully” made others struggle to discern the intended meaning of the added adverb.

In this December 2004 draft, I have removed a number unnecessary adjectives — for example I deleted a few admonitions to do things “correctly” (how else?). I also continue the campaign against redundancy, by removing “be able to” (can one imagine a standard that began “be unable to?”).

I did, however, retain those that made a mathematical point (e.g., “divide mentally,” “calculate manually,” “record systematically,” “estimate accurately”). In addition, when there were no obvious or agreed upon fixes, I let the problematic adverbs remain for the time being.

**Nouns.** In contrast to the persistent vagueness of verbs concerning students’ thoughts and actions, the *MAP K–8 Mathematics Expectations* make a special effort to define mathematical objects and relationships carefully, logically and consistently. The goal of this effort is not only to help students be clear about what words such as “ratio” or “parallel” really mean but also to illustrate throughout the K–8 program that careful definitions are an essential and distinguishing feature of mathematical thinking.

Most mathematical terms are defined in these *Expectations* without reference to students’ knowledge or performance. Instead of saying that students should “know” or “understand” or “use” the definition of parallel lines, this draft says simply that parallel lines are lines that never meet no matter how far they are extended. We simply assume that students should know, understand and be able to use the mathematics outlined in these expectations.

## Specifics

**Fractions.** A careful, methodical introduction to fractions is a distinctive feature of MAP's *K-8 Mathematics Expectations*. There is plenty of evidence that, under current practices, many students leave middle school permanently confused about fractions, ratios and percents. MAP's diagnosis is that this adult turmoil is due to confusing school interpretations that do not make mathematical sense and consequently impede understanding and encourage thoughtless memorization. In contrast, the MAP expectations concerning fractions build carefully and logically over several years, beginning with unit fractions that are interpreted as a part of a whole. Recognition that a fraction thus defined is indeed a number that can be situated on the number line follows, but only with careful explanation and numerous examples.

Critics among the MAP reviewers argue that it is a mistake to introduce fractions well before children have mastered the arithmetic of whole numbers. They claim that premature emphasis on computational proficiency with fractions leads to superficial learning that short-changes conceptual understanding. According to this view, MAP's goal of deep conceptual understanding would more likely be achieved by focusing on whole numbers in the early grades and deferring the subtleties of fractions until the middle grades.

Only comparative classroom trials can determine whether early or late introduction of fractions is better — or if it really makes any difference at all. What is most important about the MAP approach to fractions is not its timing but its logical sequencing. The same benefits of careful development would accrue if the introduction were delayed one or two years (for instance, by trading curricular space with some parts of geometry or data currently suggested for later grades).

**Number Line.** Another distinctive feature of the MAP *K-8 Mathematics Expectations* is the consistent and strong emphasis on the number line as both an anchor and metaphor for the central subject of these expectations: number. The number line provides a consistent and mathematically accurate visual representation of numbers, no matter what confusing form their symbolic expressions may take, including whole numbers, fractions, roots, mixed numbers, decimals, ratios, percents and pi. In appearance these symbols present a bewildering display of notation ( $3$ ,  $\frac{2}{5}$ ,  $\sqrt{3}$ ,  $2^{\frac{3}{7}}$ ,  $2.35$ ,  $3:5$ ,  $23.5\%$ ,  $\pi$ ). The number line unifies these varied formats in a profound mental image that will carry students well into the study of calculus and beyond.

Some reviewers expressed concern that the number line was overemphasized at the expense of other options (e.g., manipulatives) and that, in any case, it was inappropriate for the very early grades when children think of numbers as counting numbers rather than as distances along a ruled line. To respond to the latter concern, in the early grades I have used the phrase "discrete number line" to signify the image of whole numbers lined up in a row, evenly spaced, but without any continuous line joining them. The subsequent transition to the continuous line in grades 2 or 3 is straightforward.

The issue of manipulatives as possible alternatives to the number line is more complicated. Manipulatives are tools designed to help students learn mathematics and should be used whenever teachers find them appropriate. In contrast, *the number line is part of mathematics itself*. It is like a triangle or circle — an abstract object with subtle and important properties that undergirds much of what occurs in mathematics. Whereas the MAP *K–8 Mathematics Expectations* say little about manipulatives because they are primarily of pedagogical interest, the expectations say a great deal about the number line because it is a mathematical object that must be studied and understood.

**Measurement.** Some reviewers expressed concern over the location of measurement topics, namely whether they should be with Data or Geometry. Some of this concern is caused by school tradition, some by the organization of the NCTM *Standards* and some by MAP’s own shift from *Foundations for Success*, in which measurement was divided between Geometry and Data, to the *K–8 Mathematics Expectations*, in which the section on data was renamed Measurement and Data.

*In reality, there are two very different kinds of measurement topics involved in K–8 mathematics.* One, anchored in geometry, involves the logical development of general concepts of length, area and volume based on standard units and the relation of these concepts to the numerical operation of multiplication. The other, anchored in data, concerns real-world measurements with units such as inches, quarts, dollars, kilobytes or hours and involves such topics as conversion factors, measurement errors and derived measures (e.g., velocity or miles per gallon). To the extent feasible, in this revision I reintroduced that distinction (which also was present in *Foundations for Success*).

**Calculators.** Based on criticism from a few reviewers about the lack of clarity concerning the role of calculators, I inserted occasional bullets providing expectations that students should know how to use a calculator to check their work, to generate and observe numerical patterns, and occasionally to carry out complex calculations. The presence of these specific statements implies a constraint on the purpose and role of calculators in the K–8 mathematics program that should implicitly clarify MAP’s intent that other expectations be fulfilled without calculators.

Of course, this “compromise” falls short of meeting the substantive concerns of reviewers who argue that calculators are an important everyday tool whose responsible use must be explicitly taught in school mathematics programs. In a famous paper about 20 years ago, AT&T Bell Labs mathematician Henry Pollack wrote that technology not only changes *how* we teach and learn mathematics but also *what* mathematics we should be teaching and learning. The items I inserted about calculators minimally address Pollack’s first point, but to address his second point — which is what many MAP reviewers are concerned about — would require major rethinking of the mathematical priorities of *Foundations for Success*. *This I did not do.*

**Patterns.** Many reviewers criticized the May 2004 draft for undervaluing the importance of exploring patterns (numerical and geometric; increasing, decreasing, cyclic; linear, inverse, exponential; ... ). Two reasons may be advanced for this situation. One, in school mathematics, exploring patterns is more often a pedagogical device than an outcome expectation. Because the *K–8 Mathematics Expectations* focus on student outcomes rather than teacher practices, it is not surprising that particular exploration strategies used by good mathematics teachers are not visible in these expectations.

The second reason is a bit more subtle but nonetheless important to understand. The kind of reasoning involved in detecting patterns is inferential, not deductive. Mathematicians employ both kinds of reasoning in their work. However, asking for the next few terms in the sequence 1, 2, 4, 8, 16, ... is not definitively answerable since there are many logically possible extensions of this (or any) sequence. It might be 32, 64, 128, ... if the sequence represents powers of 2, or it might be 5, 10, 3, 6, ... if the sequence represents the end values of the famous “ $3n+1$ ” puzzle. Instead of expecting students to read the mind of question-writers by knowing that powers of 2 are more likely than other possibilities, MAP expects students to be able to determine and extend patterns when a rule is given (e.g., name the first 10 terms in the sequence  $f(n) = 2^{(n-1)}$ ).

### Technicalities

**Typography.** Even an inattentive reader will easily notice that the typography in this draft is rather messy. The primary reason is that Word is a terrible editor for mathematics manuscripts, so there are many “work-arounds” embedded in the text. Many of these are graphic images of examples, especially of complex fractions, that were created early in the process by an external equation editor. Because these are fixed images, they cannot be easily edited and thus have remained unchanged throughout several revisions of the manuscript.

The same should be said about illustrations. For lack of skill and appropriate tools, I have made no attempt to add illustrations (except a few routine number-line graphics that can be created from a keyboard). So the illustrations are exactly those from the first version of these *Expectations*, even though there are several different expectations or examples that now merit different and additional graphics.

A third problem, slightly more subtle, is that Word knows nothing about the conventions of mathematical typing and typesetting in which letters used as variables are set in italic to distinguish them from regular text. Changing scores of individual letters to italic is a tedious chore that is only partially implemented in this draft.

As a draft intended for editorial and mathematical review, this manuscript is not greatly handicapped by these typographical infelicities. But they do distract from understanding in any public presentation and may well cause casual readers to worry that the logic of the document is as sloppy as its typography. It would be wise to prepare a more professional presentation before any wider distribution of the document.



# MAP Mathematics Expectations

## Number & Operations

### K.1 Count objects and use numbers to express quantity.

K.1a Count up to 25 objects and tell how many there are in the counted group of objects.

**Note:** Accuracy depends on not skipping objects or counting objects twice. Counting objects foreshadows the important mathematical concept of *one-to-one correspondence*.

**Note:** Going past 20 is important to move beyond the irregular "teen" pattern into the regular twenty-one, twenty-two, ... counting routine.

K.1b Read aloud numerals 1 through 25 and match numerals with the numbers used in counting.

**Note:** In early grades "number" generally means "natural number" or, more mathematically, "non-negative integer."

K.1c Place numbers 1 through 25 in their correct sequence.

**Note:** The emphasis in kindergarten is on the sequence of numbers as discrete objects. The "number line" that displays continuous connection from one number to the next is introduced in Grade 2.

K.1d Count to 20 by twos.

- Recognize 20 as two groups of ten and as ten groups of two.

K.1e Recognize and use ordinal numbers (e.g., first, fourth, last).

**Example:** The fourth lady bug is about to fly.

### K.2 Use number notation and place value up to 20.

K.2a Understand that numbers 1 through 9 represent "ones."

K.2b Understand that numbers 11 through 19 consist of one "ten" and some "ones."

- Relate the "teen" number words to groups of objects ("ten" + some "ones").

**Example:** 13 can be called "one ten and three ones," with "thirteen" being a kind of nickname.

### K.3 Compare numbers up to 10

K.3a Compare sets of ten or fewer objects and identify which are equal to, more than, or less than others.

- Compare by matching and by counting.
- Use picture graphs (pictographs) to illustrate quantities being compared.

K.3b Recognize zero (0) as the count of "no objects."

**Note:** Zero is the answer to "how many are left?" when all of a collection of objects have been taken away.

**Example:** Zero is the number of buttons left after 7 buttons are removed from a box that contains 7 buttons.

## K.4 Understand addition as putting together and subtraction as breaking apart.

- K.4a Add and subtract single digit numbers whose total or difference is between 0 and 10.
- Write expressions such as  $5 + 2$  or  $7 - 3$  to represent situations involving sums or differences of numbers less than 10.
- K.4b Understand add as "put together" or "add onto" and solve addition problems with numbers less than 10 whose totals are less than 20.
- Understand the meaning of addition problems phrased in different ways to reflect how people actually speak.
  - Use fingers and objects to add.
  - Attach correct names to objects being added.  
**Note:** This is especially important when the objects are dissimilar. For example, the sum of 3 apples and 4 oranges is 7 fruits.
- K.4c Understand subtract as "break apart" or "take away" and solve subtraction problems using numbers between 1 and 10.
- Understand the meaning of addition problems phrased in different ways to reflect how people actually speak.  
**Example:**  $7 - 3$  equals the number of buttons left after 3 buttons are removed from a box that contains 7 buttons.
  - Recognize subtraction situations involving missing addends and comparison.
  - Use fingers, objects, and addition facts to solve subtraction problems.
- K.4d Express addition and subtraction of numbers between 1 and 10 in stories and drawings.
- Translate such stories and drawings into numerical expressions such as  $7 + 2$  or  $10 - 8$ .
  - Model, demonstrate (act out), and solve stories that illustrate addition and subtraction.

## K.5 Compose and decompose numbers 2 through 10.

- K.5a Understand that numbers greater than 2 can be decomposed in several different ways.
- Note:** Decomposition and composition of single digit numbers into other single-digit numbers is of fundamental importance to develop meaning for addition and subtraction.
- Example:**  $5 = 4 + 1 = 3 + 2$ ;  $10 = 9 + 1 = 8 + 2 = 7 + 3 = 6 + 4 = 5 + 5$ .
- Recognize 6 through 10 as "five and some ones."  
**Note:** This is an important special case because of its relation to finger counting.  
**Example:**  $6 = 5 + 1$ ;  $7 = 5 + 2$ ;  $8 = 5 + 3$ ;  $9 = 5 + 4$ ;  $10 = 5 + 5$ .

## Number & Operations

### 1.1 Understand and use number notation and place value up to 100.

- 1.1a Count to 100 by ones and tens.
- Group objects by tens and ones, and relate written numerals to counts of the groups by ones, and to counts of the groups by tens.
- 1.1b Read and write numbers up to 100 in numerals.
- Understand and use numbers up to 100 expressed orally.
  - Write numbers up to 10 in words.
- 1.1c Recognize the place value of numbers (tens, ones).
- Recognize the use of *digit* to refer to the numerals 0 through 9.
  - Arrange objects into groups of tens and ones and match the number of groups to corresponding digits in the number that represents the total count of objects.

### 1.2 Compare numbers up to 100 and arrange them in numerical order.

- 1.2a Arrange numbers in increasing and decreasing order.
- 1.2b Locate numbers up to 100 on the discrete number line.
- Understand that on the number line bigger numbers appear to the right of smaller numbers.  
**Note:** The *discrete number line* is not the continuous number line that will be used extensively in later grades, but a visual device for holding numbers in their proper regularly spaced positions. The focus in Grades K-2 is on the uniformly spaced natural numbers, not on the line that connects them. However, for simplicity, in these grades the discrete number line is often called the *number line*.
  - Use the number line to create visual representations of sequences.  
**Examples:** Even numbers, tens, multiples of five.
  - Understand and use relational words such as *equal*, *bigger*, *greater*, *greatest*, *smaller*, and *smallest*; and phrases *equal to*, *greater than*, *more than*, *less than*, and *fewer than*.
- 1.2c Compare two or more sets of objects in terms of differences in the number of elements.
- Use matching to establish a one-to-one correspondence, and count the remainder to determine the size of the difference.
  - Connect the meanings of relational terms (*bigger*, etc.) to the order of numbers, to the measurement of quantities (length, volume, weight, time), and to the operations of adding and subtracting.  
**Example:** If you add something bigger, the result is bigger, but if you take away something bigger, the result will be smaller.

### 1.3 Add, subtract, compose, and decompose numbers up to 100.

- 1.3a Be able to solve problems that require addition and subtraction of numbers up to 100 in a variety of ways.
- Know addition and subtraction facts for numbers up to 12.
  - Add and subtract efficiently, both mentally and with pencil and paper.

**Note:** Avoid sums or differences that require numbers greater than 100 or less than 0.

- Be able to explain why the method used produces the correct answer.

**Note:** Any correct method will suffice; there is no reason to insist on a particular algorithm since there are many correct methods. Common methods include "adding on" (often using fingers) and regrouping to make a ten.

**Examples:**  $6 + 8 = 6 + 4 + 4 = 10 + 4 = 14$ ; or  
 $6 + 8 = 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$ ;

- Add three single-digit numbers.  
**Examples:**  $3 + 4 + 1 = ?$ ;  $7 + 5 + 3 = ?$
- Understand and solve oral problems with a variety of phrasing, including *how many more*, or *how many fewer*.
- Know how to use a calculator to check answers.

### 1.3b Understand how to compose and decompose numbers.

- Identify and discuss patterns arising from decompositions.  
**Example:**  $8 = 7 + 1 = 1 + 7 = 6 + 2 = 2 + 6 = 5 + 3 = 3 + 5 = 4 + 4$ ;  
 $9 = 8 + 1 = 1 + 8 = 7 + 2 = 2 + 7 = 3 + 6 = 6 + 3 = 5 + 4 = 4 + 5$ .
- Represent decomposition situations using terms such as *put together*, *add to*, *take from*, *break apart*, or *compare*.

### 1.3c Use groups of tens and ones to add numbers greater than ten.

- Using objects or drawings, add the tens, add the ones, and regroup if needed.  
**Note:** Grouping relies on the commutative and associative properties of addition. Examples in early grades foreshadow more formal treatments later. The vocabulary should await later grades.

**Examples:**

(a)  $17 + 24 = 17 + 23 + 1 = 17 + 3 + 20 + 1 = 20 + 20 + 1 = 40 + 1 = 41$ .

(b)  $58 + 40 = 50 + 8 + 40 = 50 + 40 + 8 = 98$

(c)  $58 + 6 = 50 + 8 + 6 = 50 + 14 = 50 + 10 + 4 = 60 + 4 = 64$

(d)  $58 + 26 = (58 + 2) + (26 - 2) = 60 + 24 = 84$ .

### 1.3d Create and solve addition and subtraction problems with numbers smaller than 20.

- Create and discuss problems using drawings, stories, picture graphs, diagrams, symbols, and open equations (e.g.,  $4 + ? = 17$ ).
- Use the (discrete) number line to illustrate the meaning of addition and subtraction.
- Express answers in a form (verbal or numerical) that is appropriate to the original problem.
- Always check that answers are intuitively reasonable.

## Number & Operations

### 2.1 Understand and use number notation and place value up to 1000.

2.1a Count by ones, twos, fives, tens, and hundreds.

- Count accurately for at least 25 terms.

**Example:** Count by tens from 10 to 200; count by 2s from 2 to 50.

- Begin counts with numbers other than 1.

**Example:** Count by tens from 200 to 300; count by 5s from 50 to 100.

2.1b Read and write numbers up to 1,000 in numerals and in words.

- Up to 1,000, read and write numerals, understand and speak words; write words up to 100.

2.1c Recognize the place values of numbers (hundreds, tens, ones).

- Understand the role of zero in place value notation.

**Example:** In  $508 = 5$  hundreds,  $0$  tens, and  $8$  ones, the  $0$  tens cannot be ignored (even though it is equal to zero), because in place value notation it is needed to separate the hundreds position from the ones position.

**Note:** Grade 2 begins the process of numerical abstraction--of dealing with numbers beyond concrete experience. Place value, invented in ancient India, provides an efficient notation that makes this abstract process possible and comprehensible.

2.1d Understand and utilize the relative values of the different number places.

- Recognize that the hundreds place represents numbers that are ten times as large as those in the tens place, and that the units place represents numbers that are ten times smaller than those in the tens place.

**Note:** Understanding these relative values provides the foundation for understanding rounding, estimation, accuracy, and significant digits.

- Use meter sticks and related metric objects to understand how the metric system mimics the "power of ten" scaling pattern that is inherent in the place value system.

**Example:** Write lengths, as appropriate, in centimeters, decimeters, meters, and kilometers.

2.1e Compare numbers up to 1,000.

### 2.2 Locate and interpret numbers on the number line.

2.2a Recognize the continuous interpretation of the number line where points correspond to distances from the origin (zero).

- Know how to locate zero on the number line.

**Note:** The number line is an important unifying idea in mathematics. It ties together several aspects of number, including size, distance, order, positive, negative, and zero. Later it will serve as the basis for understanding rational and irrational numbers, and after that for the limit processes of calculus. In Grade 2 the interpretation of the number line advances from discrete natural numbers to a continuous line of indefinite length in both directions. Depending on context, a number  $N$  (e.g., 1 or 5) can be thought of either as a single point on the number line, or as the interval connecting the point 0 to the point  $N$ , or as the length of that interval.

2.2b Use number line pictures and manipulatives to illustrate addition and subtraction as the adding and subtracting of lengths.

**Note:** A meter stick marked in centimeters is a useful model of the number line because it reflects the place value structure of the decimal number system.

- 2.2c Understand the symbol  $\frac{1}{2}$  and the word *half* as signifying lengths and positions on the number line that are midway between two whole numbers.
- Read foot and inch rulers with uneven hash marks to the nearest half inch.

### 2.3 Add, subtract, and use numbers up to 1000.

- 2.3a Add and subtract two- and three-digit numbers with efficiency and understanding.
- Add and subtract mentally with ones, tens, and hundreds.
  - Use different ways to regroup or ungroup (decompose) to efficiently carry out addition or subtraction both mentally and with pencil and paper.
 

**Example:**  $389 + 492 = (389-8) + (8 + 492) = 381 + 500 = 881$
  - Perform calculations in writing and be able to explain reasoning to classmates and teachers.
  - Add three two-digit numbers in a single calculation.
  - Before calculating, estimate answers based on the left-most digits; after calculating use a calculator to check the answer.
- 2.3b Understand "related facts" associated with adding and subtracting.
- Note:** The expression "related facts" refers to all variations of addition and subtraction facts associated with a particular example.
- Solve addition equations with unknowns in various positions.
 

**Example:**  $348 + 486 = ?$ ,  $348 + ? = 834$ ,  $? + 486 = 834$ ,  $834 - 486 = ?$   
 $834 - ? = 348$ , and  $? - 486 = 348$ .
  - Demonstrate how carrying (in addition) and borrowing (in subtraction) relate to composing and decomposing (or grouping and ungrouping)
  - Connect the rollover cases of carrying in addition to the remote borrowing cases in subtraction.
 

**Example:**  $309 + 296 = 605$ ;  $605-296 = 309$ .
- 2.3c Create stories, make drawings, and solve problems that illustrate addition and subtraction with unknowns of various types.
- Understand situations described by phrases such as *put together* or *add to* (for addition) *take from*, *break apart*, or *compare* (for subtraction).
  - Recognize and create problems using a variety of settings and language.
 

**Caution:** Avoid being misled by (or dependent on) stock phrases such as *more* or *less* as signals for adding or subtracting.
- 2.3d Solve problems that require more than one step and that use numbers below 50.
- Note.** Since the challenge here is to deal with multi-step problems, the numbers are limited to those already mastered in the previous grade.
- Solve problems that include irrelevant information and recognize when problems do not include sufficient information to be solved.
  - Represent problems using appropriate graphical and symbolic expressions.
  - Express answers in verbal, graphical, or numerical form, using appropriate units.
  - Check results by estimation for reasonableness, and by calculator for accuracy.

## 2.4 Understand multiplication as repeated addition, and division as the inverse of multiplication.

2.4a Multiply small whole numbers by repeated addition.

- Skip count by steps of 2, 3, 4, 5, and 10 and relate patterns in these counts to multiplication.

**Example:**  $3 \times 4$  is the 3<sup>rd</sup> number in the sequence 4, 8, 12, 16, 20, ... .

- Relate multiplication by 10 to the place value system.

2.4b Understand division as the inverse of multiplication.

- Use objects to represent division of small numbers.

**Note:** As multiplication is repeated addition, so division is repeated subtraction. Consequently, division reverses the results of multiplication and *vice versa*.

**Note:** Since division is defined here as the inverse of multiplication, only certain division problems make sense, namely those that arise from a multiplication problem.

**Example:**  $8 \div 4$  is 2 since  $4 \times 2 = 8$ , but  $8 \div 3$  is not (yet) defined.

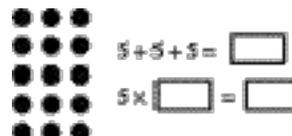
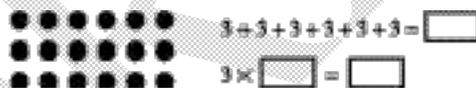
2.4c Know the multiplication table up to  $5 \times 5$ .

- Use multiplication facts within the  $5 \times 5$  table to solve related division problems.

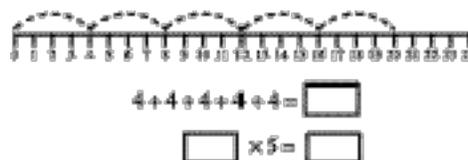
**Note:** Multiplication facts up to  $5 \times 5$  are easy to visualize in terms of objects or pictures, so introducing it in Grade 2 lays the foundation for the more complex  $10 \times 10$  multiplication expectation that is central to Grade 3.

2.4d Solve multiplication and division problems involving repeated groups and arrays of small whole numbers.

- Arrange groups of objects into rectangular arrays to illustrate repeated addition and subtraction.
- Rearrange arrays to illustrate that multiplication is commutative.



- Demonstrate skip counting on the number line and then relate this representation of repeated addition to multiplication.



## Number & Operations

### 3.1 Read, write, add, subtract, and comprehend five-digit numbers.

- 3.1a Read and write numbers up to 10,000 in numerals and in words.
- 3.1b Understand that digits in numbers represent different values depending on their location (place) in the number.
- Identify the thousands, hundreds, tens, ones positions and state what quantity each digit represents.
- Example:**  $9725 - 9325 = 400$  because  $7 - 3 = 4$  in the hundreds position.
- 3.1c Compare numbers up to 10,000.
- Understand and use the symbols  $<$ ,  $\leq$ ,  $>$ ,  $\geq$  to signify order and comparison.
  - Note especially the distinction between  $<$  and  $\leq$ , and between  $>$ ,  $\geq$ .
- Example:** There are 6 numbers that could satisfy  $97 < ? \leq 103$ , but only five that could satisfy  $97 < ? < 103$ .
- 3.1d Understand and use grouping for addition and ungrouping for subtraction.
- Recognize and use the terms *sum* and *difference*.
  - Use parentheses to signify grouping and ungrouping.
- Example:**  $375 + 726 = (3+7) \times 100 + (7+2) \times 10 + (5+6)$   
 $= 10 \times 100 + 9 \times 10 + 10 + 1 = 10 \times 100 + 10 \times 10 + 1$   
 $= (1 \times 1000) + (1 \times 100) + 0 \times 10 + 1 = 1101$
- 3.1e Add and subtract two-digit numbers mentally.
- Use a variety of methods appropriate to the problem, including adding or subtracting the smaller number by mental (or finger counting); regrouping to create tens; adding or subtracting an easier number and then compensating; creating mental pictures of manual calculation, and others.
  - Check answers with a different mental method and compare the efficiency of different methods in relation to different types of problems.
- 3.1f Judge the reasonableness of answers by estimation.
- Use highest order place value (e.g., tens or hundreds digit) to make simple estimates.
- 3.1g Solve a variety of addition and subtraction problems.
- Story problems posed both orally and in writing.
  - Problems requiring two or three separate calculations.
  - Problems that include irrelevant information.

### 3.2 Multiply and divide with numbers up to 10.

- 3.2a Understand division as an alternative way of expressing multiplication.
- Recognize and use the terms *product* and *quotient*.
  - Express a multiplication statement in terms of division, and vice versa.
- Example:**  $3 \times 8 = 24$  means that  $24 \div 3 = 8$ , and that  $24 \div 8 = 3$ .
- 3.2b Recognize different interpretations of multiplication and division and explain why they are equivalent.
- Understand multiplication as repeated addition, as area, and as the number of objects in a rectangular array.

**Example:** Compare a class with 4 rows of 9 seats, a sheet of paper that is 4 inches wide and 9 inches high, and a picnic with 4 groups of 9 children each. Contrast with a class that has 9 rows of 4 seats, a sheet of paper that is 9 inches wide and 4 inches high, and a picnic that involves 9 groups of 4 children each.

- Understand division as repeated subtraction that inverts or “undoes” multiplication.
- Understand division as representing the number of rows or columns in a rectangular array, as the number of groups resulting when a collection is partitioned into equal groups, and as the size of each such group.

**Example:** When 12 objects are partitioned into equal groups, 3 can represent either the number of groups (because 12 objects can be divided into three groups of four [4, 4, 4]) or the size of each group (because 12 objects can be divided into four groups of three [3, 3, 3, 3]).

**Note:** In early grades use only  $\div$  as the symbol for division—to avoid confusion when the slash (/) is introduced as the symbol for fractions.

### 3.2c Know the multiplication table up to $10 \times 10$ .

- Knowing the multiplication table means being able to find quickly missing values in open multiplication or division statements such as  $56 \div 8 = [ ]$ ,  $7 \times [ ] = 42$ , or  $12 \div [ ] = 4$ .

**Note:** Knowing by instant recall is the goal, but recalling patterns that enable a correct rapid response is an important early stage in achieving this skill.

### 3.3d Count aloud the first 10 multiples of each 1-digit natural number.

### 3.2e Create, analyze, and solve multiplication and division problems that involve groups and arrays.

- Describe contexts for multiplication and division facts.
- Complete sequences of multiples found in the rows and columns of multiplication tables up to 15 by 15.

### 3.2f Make comparisons that involve multiplication or division.

## 3.3 Solve contextual, experiential, and verbal problems that require several steps and more than one arithmetic operation.

**Note:** Although solving problems is implicit in every expectation (and thus often not stated explicitly), this particular standard emphasizes the important skill of employing two different arithmetical operations in a single problem.

### 3.3a Represent problems mathematically using diagrams, numbers, and symbolic expressions.

### 3.3b Express answers clearly in verbal, numerical, or graphical (bar or picture) form, using units whenever appropriate.

### 3.3c Use estimation to check answers for reasonableness, and calculators to check for accuracy.

**Note:** Problem selection should be guided by two principles: To avoid excess reliance on verbal skills, use real contexts as prompts as much as possible. And to focus on problem-solving skills, keep numbers simple, typically within the computational expectations one grade earlier.

## 3.4 Recognize negative numbers and fractions as numbers and know where they lie on the number line.

### 3.4a Know that symbols such as $-1$ , $-2$ , $-3$ represent *negative numbers* and know where they fall on the number line.

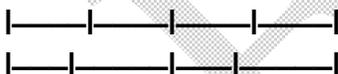
- Recognize negative numbers as part of the scale of temperature.
- Use negative numbers to count backwards below zero.
- Observe the mirror symmetry in relation to zero of positive and negative numbers.

**Caution:** In grade 3, negative numbers are introduced only as names for points to the left of zero on the number line. They are not used in arithmetic at this point (e.g., for subtraction). In particular the minus sign (-) prefix on negative numbers should not at this stage be interpreted as subtraction.

3.4b Understand that symbols such as  $\frac{1}{2}$ ,  $\frac{1}{3}$ , and  $\frac{1}{4}$  represent numbers called *unit fractions* that serve as building blocks for all fractions.

- Understand that a unit fraction represents the length of a segment that results when the *unit interval* from 0 to 1 is divided into pieces of equal length.

**Note:** A unit fraction is determined not just by the number of parts into which the unit interval is divided, but by the number of equal parts. For example, in the upper diagram that follows, each of the four line segments represents  $\frac{1}{4}$ , but in the lower diagram none represents  $\frac{1}{4}$ .

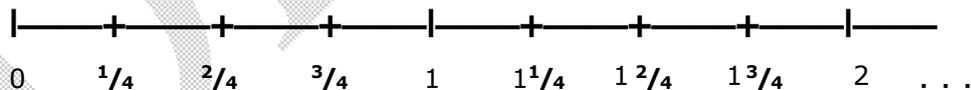


- Recognize, name, and compare unit fractions with denominators up to 10.

**Example:** The unit fraction  $\frac{1}{6}$  is smaller than the unit fraction  $\frac{1}{4}$  since when the unit interval is divided into 6 equal parts, each part is smaller than if it were divided into four equal parts. The same thing is true of cookies or pizzas: one-sixth of something is smaller than one-fourth of that same thing.

3.4c Understand that each unit fraction  $\frac{1}{n}$  generates other fractions of the form  $\frac{2}{n}$ ,  $\frac{3}{n}$ ,  $\frac{4}{n}$ , ... and know how to locate these fractions on the number line.

- Understand that  $\frac{1}{n}$  is the point to the right of 0 that demarcates the first segment created when the unit interval is divided into  $n$  equal segments. Points marking the endpoints of the other segments are labeled in succession with the numbers  $\frac{2}{n}$ ,  $\frac{3}{n}$ ,  $\frac{4}{n}$ , ... . These points represent the numbers that are called *fractions*.



- Understand that a fractional number such as  $\frac{1}{3}$  can be interpreted either as the point that lies one-third of the way from 0 to 1 on the number line or as the length of the interval between 0 and this point.

**Note:** When the unit interval is divided into  $n$  segments, the point to the right of the last ( $n^{\text{th}}$ ) segment is  $n/n$ . This point, the right end-point of the unit interval, is also the number 1.

### 3.5 Understand, interpret, and represent fractions.

3.5a Recognize and utilize different interpretations of fractions, namely, as a *point* on the number line; as a *number* that lies between two consecutive (whole) numbers; as the *length* of a segment of the real number line; and as a *part* of a whole.

**Note:** The standard of meeting this expectation is not that children be able to explain these interpretations but that they are able to use different interpretations appropriately and effectively.

3.5b Understand how a general fraction  $n/d$  is built up from  $n$  unit fractions of the form  $1/d$ .

- Understand and use the terms *numerator* and *denominator*.
- Understand that the fraction  $n/d$  is a number representing the total length of  $n$  segments created when the unit interval from 0 to 1 is divided into  $d$  equal parts.

**Note:** This definition applies even when  $n > d$  (i.e., the numerator is greater than the denominator): just lay  $n$  segments of size  $1/d$  end to end. It will produce a segment of length  $n/d$  regardless of whether  $n$  is less than, equal to, or greater than  $d$ . Consequently, there is no need to require that the numerator be smaller than the denominator.

- Recognize that when  $n = d$ , the fraction  $n/d = 1$ ; when  $n < d$ ,  $n/d < 1$ ; and when  $n > d$ ,  $n/d > 1$ .

**Examples:**  $2/2 = 1$ ,  $2/3 < 1$ , and  $3/2 > 1$ .

- Recognize the associated vocabulary of *mixed number*, *proper fraction*, and *improper fraction*.

**Note:** These terms are somewhat archaic and not of great significance. It makes no difference if the numerator of a fraction is larger than the denominator, so there is nothing "improper" about so-called "improper fractions."

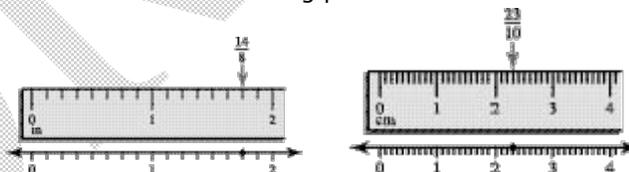
3.5c Locate fractions with denominator 2, 4, 8, and 10 on the number line.

- Understand how to interpret mixed numbers with halves and quarters (e.g.,  $3\frac{1}{2}$  or  $1\frac{1}{4}$ ) and know how to place them on the number line.

**Note:** Measurement to the nearest half or quarter inch provides a concrete model.

- Use number lines and rulers to relate fractions to whole numbers.

**Note:** The denominators 2, 4, and 8 appear on inch rulers and are created by repeatedly folding strips of paper; the denominator 10 appears on centimeter rulers and is central to understanding place value.



3.5d Understand and use the language of fractions in different contexts.

- When used alone, a fraction such as  $1/2$  is a number or a length, but when used in contexts such as " $1/2$  of an apple" the fraction represents a part of a whole.

**Note:** A similar distinction also applies to whole numbers: The phrase "I'll take 3 oranges" is not about taking the number 3, but about counting 3 oranges. Similarly, " $1/2$  of an orange" is not about the number (or unit fraction)  $1/2$ , but is a reference to a part of the whole orange.

**Note:** The vocalization of unit fractions (*one-half*, *one-third*, *one-fourth*) are expressions children will know from prior experience (e.g., one-half cup of sugar, one-quarter of an hour). Mathematical fractions extend this prior knowledge to numbers by dividing an interval of length 1. In this way, the unit fraction  $1/2$  can be defined as the number representing one-half of the unit interval.

3.5e Recognize fractions as numbers that solve division problems.

- When the unit interval is divided into equal parts to create unit fractions, the sum of all the parts adds up to the whole interval, or 1. In other words, the total of  $n$  copies of the unit fraction  $1/n$  equals 1. Since division is defined as the inverse of multiplication, this is the equivalent of saying that 1 divided by  $n$  equals  $1/n$ .

**Example:** Since 4 copies of the unit fraction  $1/4$  combine to make up the unit interval,  $4 \times (1/4) = 1$ . Equivalently,  $1 \div 4 = 1/4$ .

**Caution:** At first glance, the statement " $1 \div 4 = 1/4$ " might appear to be a tautology. It is anything but. Indeed, understanding why this innocuous equation is expressing something of importance is an important step in understanding fractions. The fraction  $1/4$  is the name of a point on the number line, the length of part of the unit interval. The open equation  $1 \div 4 = ?$  asks for a number with the property that  $4 \times ? = 1$ . By observing that the four parts of the unit interval add up to the whole interval, whose length is 1, we discover that the length of one of these parts is the unknown needed to satisfy the equation:  $4 \times 1/4 = 1$ . This justifies the assertion that  $1 \div 4 = 1/4$ .

### 3.6 Understand how to add, subtract, and compare fractions with equal denominators.

- 3.6a Recognize how adding and subtracting fractions with equal denominators can be thought of as the joining and taking away, respectively, of contiguous segments on the number line.

**Note:** Common synonyms for *equal* denominators are *common* denominators or *like* denominators or *same* denominators. The latter appear to emphasize the form of the denominator (e.g., all 4s) whereas "equal" correctly focuses on what matters, namely, the *value* of denominator.

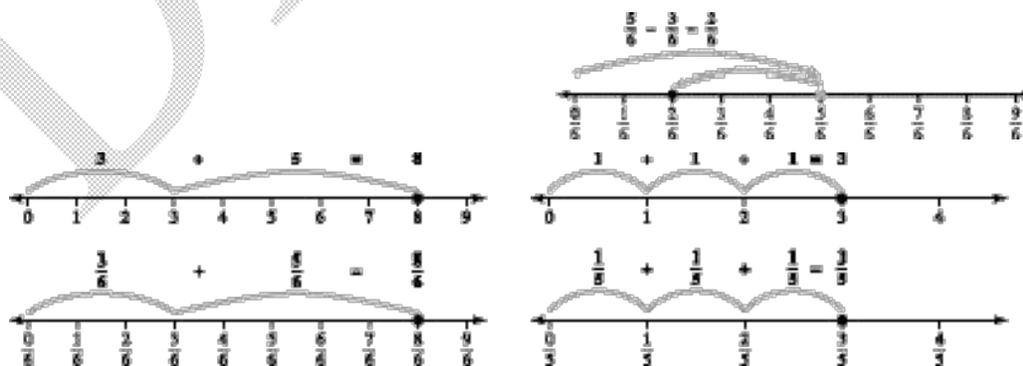
- 3.6b Understand that a fraction  $n/d$  is the sum of  $n$  unit fractions of the form  $1/d$ .

**Example:**  $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$ .

- 3.6c Compare, add, and subtract fractions with equal denominators.

- Addition and subtraction of fractions with equal denominators work exactly as do addition and subtraction of whole numbers, and therefore builds on the addition and subtraction of whole numbers.

**Note:** There is no need to simplify answers to lowest terms.



## Number & Operations

### 4.1 Read, write, add and subtract positive whole numbers.

- 4.1a Read and write numbers in numerals and in words.
- 4.1b Recognize the place values in numbers and understand what quantities each digit represents.
- Understand that each digit represents a quantity ten times as great as the digit to its right.
- 4.1c Compare natural numbers expressed in place value notation.
- 4.1d Add columns consisting of several 3-4 digit numbers.
- Use and develop skills such as creating tens and adding columns first down then up to ensure accuracy.  
**Example:** The most common example is a list of prices (e.g., a grocery bill, or a shopping list).
  - Check answers with a calculator.

### 4.2 Understand why and how to approximate or estimate.

- 4.2.a Round off numbers to the nearest 5, 10, 25, 100, or 1,000.
- *Rounding off* is something done to an overly exact number (e.g., a city's population given as 235,461). *Estimation* and *approximation* are actions taken instead of, or as a check on, an exact calculation. Estimates and approximations are almost always given as round numbers.  
**Examples:** In estimating the number of students to be served school lunch, round the number to the nearest 10 students. In estimating a town's population, rounding to the nearest 50 or 100 is generally more appropriate.
- 4.2.b Estimate answers to problems involving addition, subtraction, and multiplication.
- 4.2.c Judge the accuracy appropriate to given problems or situations.
- Use estimation to check the reasonableness of answers.
  - Pay attention to the way answers will be used to determine how much accuracy is important.  
**Note:** There are no formal rules that work in all cases. This expectation is about judgment.

### 4.3 Identify small prime and composite numbers.

- 4.3a Understand and use the definitions of prime and composite number.
- Understand and use the terms *factor* and *divisor*.
  - Apply these definitions to identify prime and composite numbers under 50.  
**Note:** A prime number is a natural number that has exactly two positive divisors, 1 and itself. A composite number is a natural number that has more than two divisors. By convention, 1 is neither prime nor composite.
- 4.3b List all factors of integers up to 50.
- 4.3c Determine if a 1-digit number is a factor of a given integer and whether a given integer is a multiple of a given 1-digit number.
- Find a common factor and a common multiple of 2 numbers.

**Note:** Common factors and multiples provide a foundation for arithmetic of fractions and for the idea of greatest common factor and least common multiple which are developed in later grades.

4.3d Recognize that some integers can be expressed as a product of factors in more than one way.

**Example:**  $12 = 4 \times 3 = 2 \times 6 = 2 \times 2 \times 3$ .

#### 4.4 Multiply small multi-digit numbers and divide by single digit numbers.

4.4a Understand and use a reliable algorithm for multiplying multi-digit numbers accurately and efficiently.

- Multiply any multi-digit number by a 1-digit number.
- Multiply a 3 digit number by a 2-digit number.
- Explain why the algorithm works.

**Example:** Justification of a multiplication algorithm relies on the distributive property applied to place value--an analysis that helps prepare students for algebra. For example, using the distributive property,  $2 \times 35$  can be written as  $2(30 + 5) = 60 + 10 = 70$ . Here's how the analysis applies to a more complex problem:  $258 \times 35$  can be written as  $(200 + 50 + 8) \times 35$ . This becomes:

$$200 \times 35 + 50 \times 35 + 8 \times 35 = 200(30 + 5) + 50(30 + 5) + 8(30 + 5).$$

From this point computations can be done mentally:

$$6000 + 1000 + 1500 + 250 + 240 + 40 = 9030.$$

4.4b Understand and use a reliable algorithm for dividing numbers by a single-digit number accurately and efficiently.

- Explain why the algorithm works.
- Understand division as fair shares and as successive subtraction, and explain how the division algorithm yields a result that conforms with these understandings.
- Check results both by multiplying and by using a calculator.

4.4c Recognize, understand, and correct common computational errors.

**Examples:** Common errors are displayed at the right.

$$\begin{array}{r} \underline{13} \text{ r } 7 \\ 6 \overline{) 85} \end{array}$$

$$\begin{array}{r} 26 \\ \times 12 \\ \hline 52 \\ + 26 \\ \hline 78 \end{array}$$

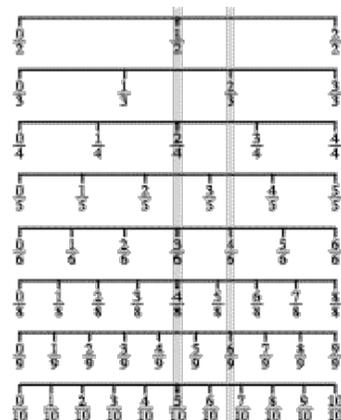
4.4d Understand the role and function of remainders in division.

- For whole numbers  $a$ ,  $b$ , and  $c$  with  $b \neq 0$ ,
  - when  $a$  is a multiple of  $b$ , the statement  $a \div b = c$  is merely a different way of writing  $a = c \times b$ ;
  - when  $a$  is not a multiple of  $b$ , the division  $a \div b$  is expressed as  $a = c \times b + r$ , where the "remainder"  $r$  is a whole number less than  $b$ .

#### 4.5 Understand and use the concept of equivalent fractions.

4.5a Understand that two fractions are equivalent if they represent the same number.

**Note:** Because equivalent fractions represent the same number, we often say, more simply, that they are the same, or equal.



**Examples:** Just as  $2 + 2$  represents the same number as 4, so  $4/6$  represents the same number as  $2/3$ . The diagram on the right shows that  $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$  and  $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ .

- Illustrate equivalent fractions using small numbers with both length and area.

**Example:** Figure 1 demonstrates in two different ways (length and area) how the fact that  $3 \times 3 = 9$  and  $3 \times 4 = 12$  makes  $3/4$  equivalent to  $9/12$ .

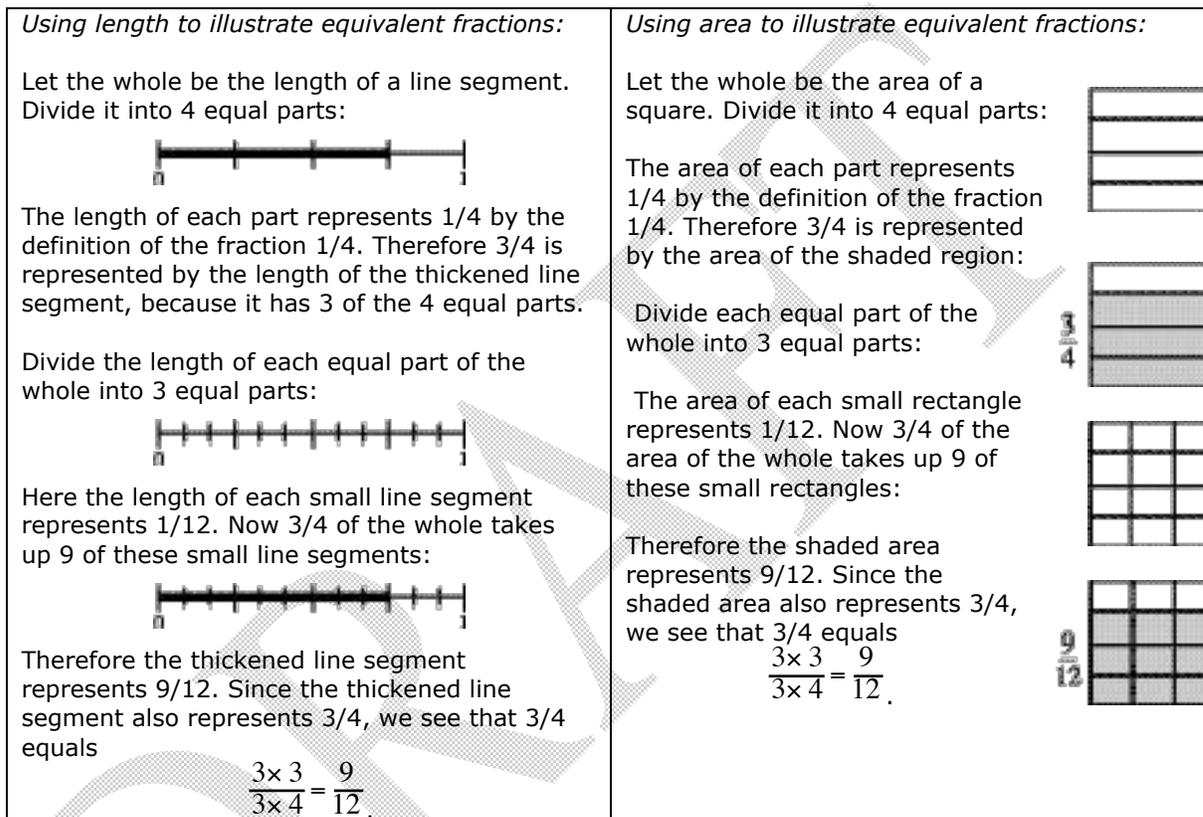


Figure 1

**Note:** Adults use three symbols interchangeably to represent division:  $\div$ ,  $/$ , and  $-$ . The latter two are also used interchangeably to represent fractions. Indeed, the symbol  $2/3$  is as often used to represent a fraction as the result of the act of division. In school, however, since fractions and division are introduced in a specific sequence, it is important that these not be used interchangeably until their equivalence has been well established and rehearsed.

4.5b Place fractions on the number line.

- Understand that equivalent fractions represent the same point on the number line.

**Note:** As introduced in Grade 3, fractions can be interpreted as a *point* on the number line; as a *number* that lies between two consecutive whole numbers; as the *length* of a segment on the real number line; and as a *part* of a whole. Two fractions are *equivalent* in each of these interpretations if they refer to the same point, number, length, or part of a whole.

#### 4.5c Understand that any two fractions can be written as equivalent fractions with equal denominators.

- Use length or area drawings to illustrate these equivalences.

**Note:** The phrase "like denominator" is often used in this context. However, it is equality, not form or "likeness," that is important.

**Example:**  $1/3$  and  $5/15$  are equivalent because both represent one-third of the unit interval. Similarly,  $1/5$  and  $3/15$  are also equivalent because both represent one-fifth of the unit interval.

**Example:**  $\frac{5}{6}$  is equivalent to  $\frac{5 \times 7}{6 \times 7}$ , and  $\frac{8}{7}$  is equivalent to  $\frac{8 \times 6}{7 \times 6}$ , both of which have the same denominators.

**Note:** More generally,  $a/b$  and  $c/d$  are equivalent to the fractions  $\frac{a \times d}{b \times d}$  and  $\frac{c \times b}{d \times b}$  respectively. This shows a general method for transforming fractions into equivalent fractions with equal (common) denominators.

**Note:** The calculations that create equivalent fractions require multiplying both the numerator and the denominator separately, by the same number. This is, of course, the same as multiplying the fraction itself by 1--which is why the two fractions are equivalent. However, it is premature at this stage to suggest that students think of  $\frac{5 \times 7}{6 \times 7}$  as  $\frac{5}{6} \times 1$  because multiplication of fractions by whole numbers is not yet addressed.

#### 4.5d Use equivalent fractions to compare fractions.

- Use the symbols  $<$  and  $>$  to make comparisons in both increasing and decreasing order.
- Emphasize fractions with denominators of 10 or less.

**Example:** The fractions  $5/6$  and  $3/8$  can be compared using the equivalent fractions  $\frac{5 \times 8}{6 \times 8}$  and  $\frac{3 \times 6}{8 \times 6}$ .

### 4.6 Add and subtract simple fractions.

#### 4.6a Add and subtract fractions by rewriting them as equivalent fractions with a common denominator.

- Solve addition and subtraction problems with fractions that are less than 1 and whose denominators are either (a) less than 10 or (b) multiples of 2 and 10, or (c) multiples of each other.
- Add and subtract lengths given as simple fractions (e.g.,  $1/3 + 1/2$  inches).
- Find the unknowns in equations such as:  $1/8 + [ ] = 5/8$  or  $3/4 - [ ] = 1/2$ .

**Note:** The idea of common denominator is a natural extension of common multiples introduced above. Addition and subtraction of fractions with common denominators was introduced in Grade 3.

**Note:** To keep calculations simple, do not use mixed numbers (e.g.,  $3\frac{1}{2}$ ) or sums involving more than two different denominators (e.g.,  $1/3 + 1/2 + 1/5$ ). Also, do not stress reduction to a 'simplest' form (because, among many reasons, such forms may not be the simplest to use in subsequent calculations).

#### 4.6b Recognize mixed numbers as an alternate notation for fractions greater than 1.

- Know how to interpret mixed numbers as an addition.
- Locate mixed numbers on the number line.

**Example:**  $\frac{23}{4} = 5\frac{3}{4}$  because on the number line  $\frac{23}{4}$  is  $\frac{3}{4}$  to the right of 5.

## 4.7 Understand and use decimal numbers up to hundredths.

4.7a Understand decimal digits in the context of place value for terminating decimals with up to two decimal places.

- A *terminating decimal* is place value notation for a special class of fractions with powers of 10 in the denominators.
- Understand the values of the digits in a decimal and express them in alternative notations.

**Examples:** The terminating decimal 0.59 equals the fraction  $59/100$ . Similarly, the decimal 12.3 is just another way of expressing the fraction  $123/10$  or the mixed number  $12\frac{3}{10}$ .

**Note:** Two-place decimals were introduced in Grade 3 to represent currency. The concept of two-place decimals as representing fractions with denominator 100 is equivalent to saying that the same amount of money can be expressed either as dollars (\$1.34) or as cents (134¢).

**Note:** The denominators of fractions associated with decimal numbers, being powers of ten, are multiples of one another. This makes adding such fractions relatively easy. For example,

$$2.34 = \frac{234}{100} = \frac{200+30+4}{100} = \frac{200}{100} + \frac{30}{100} + \frac{4}{100} = \frac{2}{1} + \frac{3}{10} + \frac{4}{100}.$$

4.7b Add and subtract decimals with up to two decimal places.

- The arithmetic of decimals becomes arithmetic of whole numbers once they are rewritten as fractions with the same denominator:

$$0.5 + 0.12 = \frac{5}{10} + \frac{12}{100} = \frac{50}{100} + \frac{12}{100} = \frac{50+12}{100} = \frac{62}{100} = .62$$

- Add and subtract two-decimal numbers, notably currency values, in vertical form.

4.7c Write tenths and hundredths in decimal and fraction notation and recognize the fraction and decimal equivalents of halves, fourths and fifths.

**Note:** "Thirds" are missing from this list since  $1/3$  cannot be represented by a terminating decimal. This is because no power of 10 is a multiple of three, so the fraction  $1/3$  does not correspond to any terminating decimal.

4.7d Use decimal notation to convert between grams and kilograms, meters and kilometers, and cents and dollars.

## 4.8 Solve multi-step problems using whole numbers, fractions, decimals, and all four arithmetic operations.

4.8a Solve problems of various types (mathematical tasks, word problems, contextual questions, and "real-world" settings) that require more than one of the four arithmetic operations.

**Note:** Problem-solving is an implied part of all expectations, but also sometimes worth special attention, as here where all four arithmetic operations are available for the first time. As noted earlier, to focus on strategies for solving problems that are cognitively more complex than those previously encountered, computational demands should be kept simple.

4.8b Understand and use parentheses to specify the order of operations.

- Know why parentheses are needed, when and how to use them, and how to evaluate expressions containing them.

- 4.8c Use the inverse relation between multiplication and division to check results when solving problems.
- Example:** Recognize that  $185 \div 5 = 39$  is wrong because  $39 \times 5 = 195$ .
- Use multiplication and addition to check the result of a division calculation that produces a non-zero remainder.
- 4.8d Translate a problem's verbal statements or contextual details into diagrams and numerical expressions, and express answers in appropriate verbal or numerical form, using units as needed.
- 4.8e Use estimation to judge the reasonableness of answers.
- 4.8f Create verbal and contextual problems representing a given number sentence and use the four operations to write number sentences for given situations.

## Number & Operations

### 5.1 Understand that every natural number can be written as a product of prime numbers in only one way (apart from order).

5.1a Extend knowledge of prime and composite numbers up to 100.

- Write composite numbers up to 100 as a product of prime factors.

**Note:** Prime and composite numbers were introduced in Grade 4. Here the goal is to investigate more examples to develop experience with larger numbers.

5.1b Decompose composite numbers into products of factors in different ways and identify which of these combinations are products of prime factors.

- Recognize that every decomposition into prime factors involves the same factors apart from order.

**Note:** It is this uniqueness ("the same factors apart from order") of the prime decomposition of integers that makes this fact important--so much so that this result is often called "the fundamental theorem of arithmetic."

**Examples:**  $24 = 2 \times 12 = 2 \times 3 \times 4 = 2 \times 3 \times 2 \times 2$ .

$$24 = 3 \times 8 = 3 \times 4 \times 2 = 3 \times 2 \times 2 \times 2.$$

$$24 = 4 \times 6 = 2 \times 2 \times 2 \times 3.$$

### 5.2 Know how to divide whole numbers.

5.2a Understand and use a reliable algorithm for division of whole numbers.

- Recognize that the division of a whole number  $a$  by a whole number  $b$  (symbolized as  $a \div b$ ) is a process to find a *quotient*  $q$  and a *remainder*  $r$  satisfying  $a = q \times b + r$ , where both  $q$  and  $r$  are whole numbers and  $r < b$ .

**Note:** The division algorithm most widely used in the United States is called *long division*. Although the term itself is often taken to mean division by a two digit number, the algorithm applies equally well to division of a multi-digit number by a single digit number.

- Understand that the long division algorithm is a repeated application of division-with-remainder.

**Example:** To divide 85 by 6, write  $85 = 80 + 5$ . Dividing 80 by 6 yields 6 10s with 20 left over. In other words,  $85 = 80 + 5 = (10 \times 6) + 20 + 5 = (10 \times 6) + 25$ . In the long division algorithm, this is written as 6 in the tens place with a remainder of 25. Next, in long division, we divide the remainder 25 by 6:  $25 = (4 \times 6) + 1$ . Combining both steps yields  $85 = (10 \times 6) + 25 = (10 \times 6) + (4 \times 6) + 1 = (14 \times 6) + 1$ .

**Note:** Since long division is a process in which the same steps are repeated until an answer is obtained, the example just given offers sufficient understanding of the general process.

$$\begin{array}{r} 14 \\ 6 \overline{) 85} \\ \underline{- 60} \\ 25 \\ \underline{- 24} \\ 1 \end{array}$$

5.2b Divide numbers up to 1,000 by numbers up to 100 using long division or some comparable approach.

- Estimate accurately in the steps of the long division algorithm.

**Example:** To compute  $6512 \div 27$  requires knowing how many 27's there are in 65, in 111, and in 32.

- Check results by verifying the division equation  $a = q \times b + r$ , both manually and with a calculator.

5.2c Know and use mental methods to calculate or estimate the answers to division problems.

- Mentally divide numbers by ten, one hundred, and one thousand.

- Where possible, break apart numbers before dividing to simplify mental calculations.

**Example:** Divide 49 by 4 by writing  $49 = 48 + 1$ . Since  $48/4 = 12$ ,  $49 \div 4 =$  yields the quotient 12 and remainder 1.

### 5.3 Understand how to add and subtract fractions.

- 5.3a Add fractions with unequal denominators by rewriting them as equivalent fractions with equal denominators.

**Note:** In Grade 4, addition of fractions was restricted to unit fractions, or to those in which one denominator was a multiple of the other. In both cases, these restrictions simplify the required calculations. Here the goal is to understand and learn to do the most general case.

- Understand and use the general formula  $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ .

**Note:** There is no need to find a *least* common denominator. The easiest common denominator of  $\frac{a}{b}$  and  $\frac{c}{d}$  is most often  $bd$ .

- When necessary, use calculators to carry out the required multiplications.

**Example:**  $17/19 + 13/14 = [(17 \times 14) + (13 \times 19)]/(19 \times 14)$   
 $= (238 + 247)/266 = 485/266$

- 5.3b Add and subtract mixed numbers.

**Example:**  $2\frac{1}{15} - 1\frac{3}{4} = \frac{30}{15} + \frac{1}{15} - \left(\frac{4}{4} + \frac{3}{4}\right) = \frac{31}{15} - \frac{7}{4} = \frac{124}{60} - \frac{105}{60} = \frac{19}{60}$

- 5.3c Find the unknown in simple equations involving fractions and mixed numbers.

**Examples:**  $2\frac{2}{3} + [ ] = 5\frac{1}{4}$ ;  $[ ] \times 14 + 3 = 101$ .

### 5.4 Understand what it means to multiply fractions and know how to do it.

- 5.4a Understand how multiplying a fraction by a whole number can be interpreted as repeated addition of the fraction.

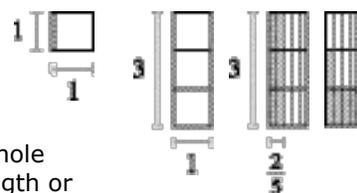
**Example:**  $3 \times \frac{2}{5}$  can be thought of as  $\frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{6}{5}$ .

**Note:** As introduced in Grade 3, fractions can be interpreted as a *point* on the number line; as a *number* that lies between two consecutive whole numbers; as the *length* of a segment on the real number line; and as a *part* of a whole. Defining multiplication of fractions by whole numbers as repeated addition is analogous to the how multiplication of whole numbers is understood, and readily conforms to the number and length interpretations of fractions.

- In general, if  $a$ ,  $b$ , and  $c$  are whole numbers and  $c \neq 0$  then  $a \times \frac{b}{c} = \frac{ab}{c}$ .

**Note:** In interpreting multiplication of a fraction by a whole number as repeated addition we introduce a curious asymmetry.  $3 \times \frac{2}{5}$  is  $\frac{2}{5}$  added to itself three times, but it does not make sense to think of  $\frac{2}{5} \times 3$  as 3 being added to itself  $\frac{2}{5}$  times. This leaves  $\frac{2}{5} \times 3$  undefined under this interpretation. If we were sure that multiplication of fractions is commutative, as is multiplication of whole numbers, then we would be able to say that  $\frac{2}{5} \times 3 = 3 \times \frac{2}{5}$ . But to do this requires the "part of a whole" interpretation of fractions.

**Example:** The multiplication of a fraction by a whole number can also be interpreted by means of a length or



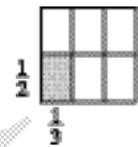
area model. Here's an example of how using an area model for  $3 \times \frac{2}{5}$ . Taking the whole as the area of a unit square,  $3 \times 1$  would be the area of the tower consisting of three unit squares. Since  $\frac{2}{5}$  means dividing the whole into 5 equal parts and taking 2 of them, the sum  $\frac{2}{5} + \frac{2}{5} + \frac{2}{5}$  can be represented by the shaded area in the middle figure on the right. The figure on the far right rearranges the small shaded rectangles to show that  $3 \times \frac{2}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{6}{5}$ .

5.4b Understand how multiplying two fractions can be interpreted in terms of an area model.

- Understand why the product of two unit fractions is a unit fraction whose denominator is the product of the denominators of the two unit fractions.
 

**Note:** Taking the whole to be a unit square, then  $\frac{1}{a} \times \frac{1}{b}$  is by definition the area of a rectangle with length  $\frac{1}{a}$  and width  $\frac{1}{b}$ . In symbols,  $\frac{1}{a} \times \frac{1}{b} = \frac{1}{ab}$ .

**Example:** Let the whole be the area of a unit square. Then  $\frac{1}{2} \times \frac{1}{3}$  is by definition the area of a rectangle with sides of length  $\frac{1}{2}$  and  $\frac{1}{3}$ . The shaded rectangle in this drawing of the unit square is such a rectangle. The shaded area is also  $\frac{1}{6}$  of the area of the whole. Therefore,  $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ .

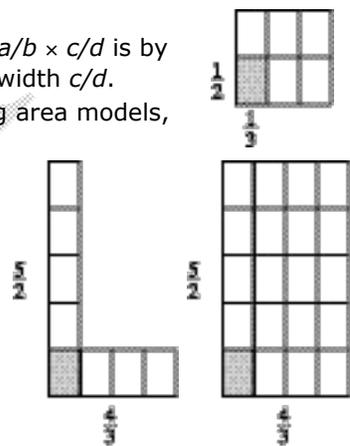


- Interpret the formula  $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$  in terms of area.

**Note:** If the area of the whole is a unit square, then  $\frac{a}{b} \times \frac{c}{d}$  is by definition the area of a rectangle with length  $\frac{a}{b}$  and width  $\frac{c}{d}$ .

**Example:** To illustrate the multiplication  $\frac{5}{2} \times \frac{4}{3}$  using area models,

let the whole be the area of a unit square. Then  $\frac{1}{2} \times \frac{1}{3}$  is the area of the shaded rectangle with length of side  $\frac{1}{2}$  and width  $\frac{1}{3}$ . By definition,  $\frac{5}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$  and  $\frac{4}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ . These are illustrated in the diagrams on the right. The large rectangle has been made from  $5 \times 4$  copies of the small shaded rectangle shown above. Since  $\frac{1}{2}$  and  $\frac{1}{3}$  are unit fractions, the area of the shaded rectangle is  $\frac{1}{2} \times \frac{1}{3} = \frac{1}{(2 \times 3)}$ . Therefore the area of the large rectangle is  $(5 \times 4) / (2 \times 3)$ .



- Recognize that the formula  $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$  shows that the multiplication of fractions is commutative.

**Note:** By validating commutativity of multiplication, the area model provides the crucial feature that is missing from the "repeated addition" model for multiplication of fractions. This shows that  $\frac{2}{5} \times 3 = 3 \times \frac{2}{5}$ .

**Note:** The formula for multiplying fractions can be used to show that fractions also obey the associative and distributive laws of whole number arithmetic. Experience with examples is sufficient to gain insight into just how this works.

5.4c Understand why " $\frac{a}{b}$  of  $c$ " is the same as " $\frac{a}{b} \times c$ ."

**Example:** The phrase " $\frac{2}{5}$  of 3" mean  $\frac{2}{5}$  of a whole that is 3 units (e.g.,  $\frac{2}{5}$  of \$3,  $\frac{2}{5}$  of 3 pizzas,  $\frac{2}{5}$  of 3 cps of sugar). To take  $\frac{2}{5}$  of 3 units, take  $\frac{2}{5}$  of each unit and add them together:  $\frac{2}{5} + \frac{2}{5} + \frac{2}{5} = 3 \times \frac{2}{5}$ . Since multiplication of fractions is commutative,  $3 \times \frac{2}{5} = \frac{2}{5} \times 3$ .

**Example:**  $\frac{3}{4}$  of the length of a 12-inch ruler is 9 inches, while  $\frac{3}{4}$  of the length of 100-centimeter ruler is 75 centimeters.

5.4d Understand that the product of a positive number with a positive fraction less than 1 is smaller than the original number.

- In symbols, if  $a$ ,  $b$ ,  $c$ , and  $d$  are all  $> 0$  and  $\frac{a}{b} < 1$  then  $\frac{a}{b} \times \frac{c}{d} < \frac{c}{d}$ .

**Note:** Area is again the easiest model:  $a/b \times c/d$  can be represented by a rectangle with dimensions  $a/b$  and  $c/d$ , whereas  $c/d$  can be represented by a rectangle of dimensions 1 and  $c/d$ . When  $a/b < 1$ , the former will fit inside the latter, thus showing that it has smaller area.

## 5.5 Understand and use the interpretation of a fraction as division.

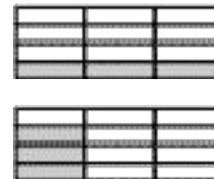
5.5a Understand why the fraction  $a/b$  can be considered an answer to the division problem  $a \div b$ .

- Among whole numbers, the answer to  $a \div b$  is a quotient and a remainder (which may be zero). Among fractions, the answer to  $a \div b$  is the fraction  $a/b$ .

**Note:** The expression  $a \div b$  where  $a$  and  $b$  are whole numbers signifies a process to find a *quotient*  $q$  and a *remainder*  $r$  satisfying  $a = q \times b + r$ , where both  $q$  and  $r$  are whole numbers and  $r < b$ . If we permit fractions as answers, then  $q = a/b$  and  $r = 0$  will always solve the division problem since  $a = a/b \times b + 0$ .

**Note:** This fact justifies using the fraction bar ( $\frac{\quad}{\quad}$  or  $/$ ) to denote division rather than the division symbol ( $\div$ ). Beyond elementary school, this is the common convention since the limitation of integer answers (quotient and remainder) is much less common.

**Example:** To illustrate the assertion that  $a/b = a \div b$  with the fraction  $3/4$ , begin as usual with the whole being a unit square.  $3 \div 4$  is the area of one part when three wholes are divided into 4 equal parts as shown. By moving all three shaded rectangles into the same whole, as shown, they form 3 parts of a whole that has been divided into 4 equal parts. That is the definition of the fraction  $3/4$ . Thus  $3 \div 4 = 3/4$ .



**Note:** Another way to think about the relation between fractions and divisions is to begin with  $4 \times 3/4 = 3$ . This says that 4 equal parts, each of size  $3/4$ , make up 3 wholes. Therefore,  $3/4$  is one part when 3 is divided into 4 equal parts--which is one interpretation of  $3 \div 4$ . (This latter interpretation of division is often called "equal shares" or "partitive.")

5.5b Understand how to divide a fraction by a fraction and to solve related problems.

- As with whole numbers, division of fractions is just a different way to write multiplication: if  $A$ ,  $B$ , and  $C$  are fractions with  $B \neq 0$ , then  $A/B = C$  means  $A = C \cdot B$

**Note:** The dot ( $\cdot$ ) is an alternative to the cross ( $\times$ ) as a notation for multiplication. (Computers generally use the asterisk ( $*$ ) in place of a dot.) In written mathematics, but never on a computer, the dot is often omitted (e.g.,  $ab$  means  $a \cdot b$ ). As students move beyond the arithmetic of whole numbers to the arithmetic of fractions and decimals, the symbols  $\cdot$  and  $/$  tend to replace  $\times$  and  $\div$ .

- Divide a fraction  $a/b$  (where  $b \neq 0$ ) by a non-zero whole number  $c$ : because  $\frac{a}{b} = \frac{a}{bc} \times c$ , this division follows the rule  $\frac{a/b}{c} = \frac{a}{bc}$ .

**Example:**  $\frac{6/7}{4} = \frac{6}{4 \times 7}$  because  $\frac{6}{7} = \frac{6}{4 \times 7} \times 4$ . In the partitive interpretation of division,  $\frac{6/7}{4}$  is one part in a division of  $\frac{6}{7}$  into 4 equal parts.

- Divide a whole number  $a$  by a unit fraction  $1/b$  ( $b \neq 0$ ): because  $a = ab \times \frac{1}{b}$ , this division follows the rule  $a / (\frac{1}{b}) = ab$ .

**Example:** Because  $5 = (5 \times 6) \times \frac{1}{6}$ ,  $5 / (\frac{1}{6}) = 5 \times 6$ . In the measurement sense of division,  $5 / (\frac{1}{6}) = 5 \times 6$  is the answer to the question "how many parts of size  $\frac{1}{6}$  can 5 be divided into?" Since there are 6 parts of size  $\frac{1}{6}$  in one whole, there are  $5 \times 6$  parts of size  $\frac{1}{6}$  in 5 wholes.

### 5.5c Express division with remainder in the form of mixed numbers.

- When a division problem  $a \div b$  is resolved into a quotient  $q$  and a remainder  $r$ , then  $a = q \times b + r$ . It follows that  $a/b$  equals the fraction  $(qb + r)/b$ , which in turn equals the mixed number  $= q\frac{r}{b}$ .

**Example:**  $\frac{37}{7} = \frac{(5 \times 7) + 2}{7} = \frac{(5 \times 7)}{7} + \frac{2}{7} = 5 + \frac{2}{7}$ , which is equal to  $5\frac{2}{7}$  by definition.

**Note:** Fractions greater than 1 are often called *improper* fractions, although there is no justification nor need for this label.

### 5.5d Understand division as the inverse of multiplication, and *vice versa*.

**Note:** Division was defined in Grade 2 as an action that reverses the results of multiplication. At that time, using only integers, division was limited to composite numbers and their factors (e.g.,  $6 \div 3$ , but not  $6 \div 4$ ). Only now, using fractions as well as whole numbers, can this inverse relationship be fully understood.

**Note:** Although in previous grades the word number meant positive whole number, hereafter it will generally mean positive fraction, which encompasses all whole and mixed numbers.

- For any numbers  $a$  and  $b$  with  $b \neq 0$ ,  $(a \times b) \div b = a$  and  $(a \div b) \times b = a$ . In words, if a number (fraction)  $a$  is first multiplied by  $b$  and then divided by  $b$ , the result is the original number  $a$ , and the same is true if we first divide and then multiply.

## 5.6 Understand how to multiply terminating decimals by whole numbers.

### 5.6a Multiplying a terminating decimal by a whole number is equivalent to multiplying a fraction by a whole number.

**Example:**  $7.53 \times 5 = (753/100) \times 5 = (753 \times 5) / 100 = 3765/100 = 37.65$ .

### 5.6b Understand how to place the decimal point in an answer to a multiplication problem both by estimation and by calculation.

**Example.**  $5 \times 0.79 = 3.95$  because  $5 \times \frac{79}{100} = \frac{5 \times 79}{100} = \frac{395}{100} = 3.95$ . This can easily be estimated because 0.79 is less than 1, so  $5 \times 0.79$  must be less than 5. Therefore the answer cannot be 395.0 or 39.5. Similarly, since  $5 > 1$ ,  $5 \times 0.79$  must be greater than .79, so the answer cannot be .395. Thus it must be 3.95.

- When a number is multiplied by a power of ten, the place value of the digits in the number are increased according to the power of ten; the reverse happens when a number is divided by a power of ten.

**Note:** As a consequence, when multiplying a whole number by 10, 100, 1,000, the decimal point shifts to the right by 1, 2, or 3 places. Similarly, when dividing a whole number by 10, 100, 1,000, the decimal point shifts to the left.

### 5.6c Demonstrate with examples that multiplication of a number by a decimal or a fraction may result in either a smaller or a larger number.

**Note:** Decimals, like fractions, can be greater than one.

## 5.7 Understand the notation and calculation of positive whole number powers.

5.7a Recognize and use the definition and notation for exponents.

- If  $p$  is a positive whole number, then  $a^p$  means  $a \times a \times a \times \dots \times a$  ( $p$  times).

**Note:** Emphasize two special cases: powers of 2 and powers of 10.

- Understand and use the language of exponents and powers.

**Note:** In the expression  $10^3$ , 3 is an *exponent*, and  $10^3$  is a *power* of 10.

## 5.8 Solve multi-step problems using multi-digit positive numbers, fractions, and decimals.

5.8a Solve problems of various types--mathematical tasks, word problems, contextual questions, and "real-world" settings.

**Note:** As noted earlier, problem-solving is an implied part of all expectations. To focus on strategies for solving problems that are cognitively more complex than those previously encountered, computational demands should be kept simple.

5.8b Translate a problem's verbal statements or contextual details into diagrams, symbols, and numerical expressions.

5.8c Express answers in appropriate verbal or numerical form.

- Provide units in answers.
- Use estimation to judge reasonableness of answers.
- Use calculators to check computations.
- Round off answers as needed to a reasonable number of decimal places.

5.8d Solve problems that require a mixture of arithmetic operations, parentheses, and arithmetic laws (commutative, distributive, associative).

5.8e Use mental arithmetic with simple multiplication and division of whole numbers, fractions, and decimals.

DRAFT

## Number & Operations

### 6.1 Understand and use negative numbers.

6.1a Know the definition of a negative number and how to locate negative numbers on the number line.

- If  $a$  is a positive number,  $-a$  is a number that satisfies  $a + (-a) = 0$ .
- On the number line,  $-a$  is the mirror image of  $a$  with respect to 0; it lies as far to the left of 0 as  $a$  lies to the right.

**Note:** In elementary school, a negative number  $-a$  is sometimes called the "opposite" of  $a$ , but this terminology is not used in later grades.

- Negative numbers may be either whole numbers or fractions.
- The positive whole numbers together with their negative counterparts and zero are called *integers*.

**Note:** The properties of negative numbers apply equally to integers and to fractions. Thus it is just as effective (and certainly easier) to limit to integers all examples that introduce the behavior of negative numbers.

**Note:** The positive fractions together with the negative fractions and zero (which include all integers) are called *rational numbers*. In grade 6, these are all the numbers we have, so they are usually referred to just as "numbers." Later when irrational numbers are introduced, the distinction between rational and irrational will be important--but not now.

6.1b Understand why  $-(-a) = a$  for any number  $a$ , both when  $a$  is positive and when  $a$  is negative.

- Use parentheses as in  $-(-a)$  to distinguish the subtraction operation (minus) from the negative symbol.

6.1c Use the number line to demonstrate how to subtract a larger number from a smaller one.

- If  $b > a$ , the point  $c$  on the number line that lies at distance  $b-a$  to the left of zero satisfies the relation  $a - b = c$ . Thus  $a - b = -(b-a)$ .
- Subtracting a smaller from a larger number is the same as adding the negative of the smaller number to the larger. That is, if  $a > b$  then  $a - b = a + (-b)$ .

**Note:** Formally,  $a + (-b) = (a - b) + b + (-b) = (a - b) + 0 = a - b$ .

- Recognize that  $a + (-b) = a - b$  (even when  $b > a$ ).

**Example:**  $3 - 8 = -5$  because  $5 + (3 - 8) = 5 + (3 + (-8)) = (5 + 3) + (-8) = 8 + (-8) = 0$ . Therefore,  $3 - 8$  satisfies the definition of  $-5$  as being that number which, when added to 5, yields zero.

6.1d Recognize that all numbers, positive and negative, satisfy the same commutative, associative and distributive laws.

**Note:** Demonstrations of these laws are part of Algebra, below. Here recognition and fluent use are the important issues.

### 6.2 Understand how to divide positive fractions and mixed numbers.

6.2a. Understand that division of fractions has the same meaning as does division of whole numbers.

- Just as  $A/B = C$  means that  $A = C \times B$  for whole numbers, so  $(a/b) / (c/d) = M/N$  means that  $(a/b) = (M/N) \times (c/d)$ .

6.2b Use and be able to explain the "invert and multiply" rule for division of fractions.

- Invert and multiply means:  $(a/b)/(c/d) = (a/b) \times (d/c) = (ad)/(bc)$ .  
**Note:** To verify that  $(a/b)/(c/d) = (ad)/(bc)$ , we need to check that  $(ad)/(bc)$  satisfies the definition of  $(a/b)/(c/d)$ , namely, that  $(a/b) = (ad)/(bc) \times (c/d)$ :  
 $(ad)/(bc) \times (c/d) = (a/b) \times (d/c) \times (c/d) = (a/b) \times 1 = a/b$ .

6.2c Find unknowns in division and multiplication problems using both whole and mixed numbers.

- Solve problems of the form  $a \div [] = b$ ,  $a \times [] = b$ ,  $[] \div a = b$ .  
**Examples:**  $1/4 \div [] = 1$ ;  $1/4 \div [] = 3/4$ ;  $1/2 = 1 \times []$ ;  $2^{1/3} \div 1^{1/2} = []$ ;  $2^{1/3} \div [] = 1^{1/2}$ .

6.2d Create and solve contextual problems that lead naturally to division of fractions.

- Recognize that division by a unit fraction  $1/n$  is the same as multiplying by its denominator  $n$ .

### 6.3 Understand and use ratios and percentages.

6.3a Understand ratio as a fraction used to compare two quantities by division.

- Recognize  $a:b$  and  $a/b$  as alternative notations for ratios.  
**Note:** A ratio is often thought of as a pair of numbers rather than as a single number. Two such pairs of numbers represent the same ratio if one is a non-zero multiple of the other or equivalently, if when interpreted as fractions, they are equivalent.  
**Example:** 2:4 is the same ratio as 6:12, 8:16, or 1:2.
- Understand that quantities  $a$  and  $b$  can be compared using either subtraction ( $a-b$ ) or division ( $a/b$ ).
- Recognize that the terms *numerator* and *denominator* apply to ratios just as they do to fractions.

6.3b Understand that percentage is a standardized ratio with denominator 100.

- Recognize common percentages and ratios based on fractions whose denominators are 2, 3, 4, 5 or 10.  
**Examples:** 20%, 25%,  $33\frac{1}{3}\%$ , 40%, 50%,  $66\frac{2}{3}\%$ , 90%, and 100% and their ratio, fraction, and decimal equivalents.
- Express the ratio between two quantities as a percent, and a percent as a ratio or fraction.

6.3c Create and solve word problems involving ratio and percentage.

- Write number sentences and contextual problems involving ratio and percentage.

### 6.4 Understand and use exponents and scientific notation.

6.4a Calculate with integers using the law of exponents:  $b^n \times b^m = b^{n+m}$ .

- Just as  $b \times n$  (with  $b$  and  $n$  positive) can be understood as  $b$  added to itself  $n$  times, so  $b^n$  can be understood as  $b$  multiplied by itself  $n$  times.  
**Note:** The law of exponents for positive exponents is just a restatement of this definition, since both  $b^n \times b^m$  and  $b^{n+m}$  mean  $b$  multiplied by itself  $n+m$  times.  
**Note:** If  $b > 0$ ,  $b^1 = b$ ,  $b^0 = 1$ . The same is true of  $b < 0$ . If  $b=0$ ,  $0^1=0$ , but  $0^0$  is not defined.
- If  $n > 0$ ,  $b^{-n}$  means  $1/b^n$  (that is, 1 divided by  $b$   $n$  times).

**Note:** This definition of  $b^{-n}$  is designed to make the law of exponents work for all integers (positive or negative):  $b^n \times b^{-n}$  means  $b$  multiplied by itself  $n$  times, then divided by  $b$   $n$  times, yielding 1. Thus  $b^n \times b^{-n} = b^{n+(-n)} = b^0 = 1$ .

**Example:**  $3^{-2} = (1/3)^2 = 1/9$ ;  $3^3 = 27$ ;  $3^3 \times 3^{-2} = 3^{(3-2)} = 3^1 = 3 = 27 \times 1/9 = 27/9$ .

- 6.4b Understand scientific notation and use it to express numbers and to compute products and quotients.
- Recognize the importance of scientific notation to express very large and very small numbers.
  - Locate very large and very small numbers on the number line.
  - Understand the concept of *significant digit* and the role of scientific notation in expressing both magnitude and degree of accuracy.
- 6.4c Model exponential behavior with contextual illustrations based on population growth and compound interest.

## 6.5 Solve multi-step mathematical, contextual and verbal problems using rational numbers.

- 6.5a Solve arithmetic problems involving more than one arithmetic operation using rational numbers.
- Calculate with and solve problems involving negative numbers, percentages, ratios, exponents, and scientific notation.  
**Note:** As usual, keep calculations simple in order to focus on the new concepts.
  - Compare numbers expressed in different ways and locate them on the number line.
  - Write number sentences involving negative numbers, percentages, ratios, exponents, and scientific notation.
- 6.5b Solve relevant contextual problems (e.g., sports, discounts, sales tax, simple and compound interest).
- Represent problems mathematically using diagrams, numbers, and symbolic expressions.
  - Express answers clearly in verbal, numerical, symbolic, or graphical form.
  - Use estimation to check answers for reasonableness and calculators to check for accuracy.
  - Describe real situations that require understanding of and calculation with negative numbers, percentages, ratios, exponents, and scientific notation.

## Number & Operations

### 7.1 Understand and work with the system of rational numbers.

#### 7.1a Understand the meaning of rational numbers.

- Rational numbers are numbers that can be expressed as a ratio of integers.
- Whole numbers are rational numbers since they can be expressed as a ratio with the integer 1 in the denominator.
- Mixed numbers are rational numbers since they can be expressed as a so-called "improper" fraction.
- Finite (terminating) decimal numbers are rational numbers.
- Negative fractions are rational numbers since they can be expressed as the ratio of a negative and a positive integer.

**Note:** Although this may seem self-evident, it is actually rather subtle. The rational number  $-12/7$  is not defined as a fraction, but as the "opposite" of  $12/7$ , i.e.,  $-12/7 + 12/7 = 0$ . As a fraction, the positive rational number  $12/7$  twelve one-sevenths of a whole. But no similar interpretation makes sense for  $-12/7$ . Nonetheless, we can show that  $-12/7$  is indeed the quotient of  $-12$  divided by  $7$  (just as it appears). Since  $7 \times (-12/7)$  means  $-12/7$  added to itself 7 times, it can be represented as  $-(\frac{12}{7} + \frac{12}{7} + \frac{12}{7} + \frac{12}{7} + \frac{12}{7} + \frac{12}{7} + \frac{12}{7}) = -((7 \times 12)/7) = -12$ .

Therefore, if  $n = -12/7$ ,  $7 \times n = -12$ . This means, by the definition of division, that the rational number  $n$  is the fraction  $-12/7$ .

- Percents can be thought of as rational numbers since they are ratios with denominators equal to 100.

#### 7.1b Locate rational numbers on the number line.

- Recognize that a number and its negative are mirror images with respect to the 0.
- Understand that between every two rational numbers, no matter how close together, there are many others.

**Note:** One way to find some is to average the two given numbers, and then average the averages.

**Note:** The visual image of the number line provides an anchor for understanding numerical size and order that is essential to the development of arithmetic intuition.

#### 7.1c Understand the properties of rational numbers.

- Adding, subtracting, multiplying, and dividing rational numbers always produces another rational number.
- The sum and product of two rational numbers (positive or negative, integer or fraction) satisfies the commutative, associative and distributive laws.
- If  $r$  is a rational number, then  $r+0 = r$  and  $r \times 1 = r$ .
- If  $r$  and  $s$  are a rational numbers with  $r+s = 0$ , then  $s = -r$  (the *negative* of  $r$ ).
- If  $r$  and  $s$  are a rational numbers with  $r \times s = 1$  and  $r \neq 0$ , then  $s = 1/r$  (the *reciprocal* of  $r$ ).

#### 7.1d Understand and use standard rules for inequalities when comparing rational numbers.

- For any rational numbers  $a$ ,  $b$ , and  $c$ ,
  - $a < b$  implies  $a + c < b + c$ ;
  - $a < b$  implies  $-a > -b$ ;
  - if  $c > 0$ ,  $a < b$  implies  $ac < bc$ ;

-- if  $c < 0$ ,  $a < b$  implies  $ac < bc$ ;

**Note:** Numerical and number line examples are more important than literal formulas as demonstrations of these rules.

7.1e Understand and work with greatest common divisors and least common multiples.

- Recognize common abbreviations such as gcd and lcm.
- Use greatest common divisors to reduce fractions  $n/m$  and ratios  $n:m$  to an equivalent form in which  $\text{gcd}(n,m) = 1$ . Fractions in which  $\text{gcd}(n,m) = 1$  are said to be in *lowest terms*.

7.1f Employ effective methods of calculation with rational numbers.

- Transform numbers from one form to another (fractions, decimals, percents, mixed numbers) so as to permit efficient calculation.
- Make judicious use of calculators or computers.
- Recognize the phenomenon of roundoff error and compensate for inaccuracies it introduces.

**Note:** Much of the early manipulation in each problem should be performed manually, reserving calculator use to the final stage where a complex answer may need to be converted into decimal form. If a calculator is used too early, important information about the numerical character of the final answer may be lost to early roundoff error.

- Check answers both by estimation and by appropriate independent calculations.

**Examples:** Multiply to check division; use decimals to check calculations with fractions; add a column of numbers up to check addition first done from top down.

- Use calculators to check manual computations, and use estimation to check calculator answers.

## 7.2 Understand methods for converting between fraction and decimal forms of rational numbers.

7.2a Understand how to convert a rational number into a decimal.

- The decimal form of a rational number either terminates or eventually repeats.

**Note:** The remainder in long division is always less than the divisor. So if it does not terminate, the division algorithm will eventually return some remainder for a second time. Thereafter, the process will repeat *ad infinitum*.

- A fraction has a terminating decimal expansion if and only if its denominator in reduced form has only 2s and 5s as factors.

**Note:** This assertion is an example of a mathematical statement of considerable significance--the "if and only if" condition--that will be appear often in Grades 7 and 8.

**Note:** The *reduced form* of a fraction is an equivalent fraction in which no factor is common to both the numerator and denominator.

**Examples:**  $15.4 = 154/10 = 77/5$ ;  $2.35 = 235/100 = 47/20$ .

7.2b Understand how to convert a finite decimal into a rational number.

- A terminating decimal equals a fraction with a denominator of the form  $10^n$  where  $n$  is the number of digits in the number to the right of the decimal point.

**Note:** Typically this fraction is reduced to an equivalent form by eliminating factors that are common to both the numerator and denominator (e.g.,  $.68 = 68/100 = 17/25$ ).

**Note:** Converting a repeating decimal to a fraction requires a bit of algebra, so is better left to a later grade.

### 7.2c Know divisibility rules and use them to help factor numbers.

- If  $N$  is a positive integer, and
  - if the last digit is even,  $N$  is divisible by 2;
  - if the sum of the digits of  $N$  is divisible by 3, so is  $N$ ;
  - if the last two digits of  $N$  form a number divisible by 4, so is  $N$ ;
  - if the last digit of  $N$  is a 5 or a 0,  $N$  is divisible by 5;
  - if the sum of the digits of  $N$  is divisible by 9,  $N$  is also; and
  - if the last digit of  $N$  is 0,  $N$  is divisible by 10.

**Example:** 165 is divisible by both 3 (since  $1+6+5 = 12$  is divisible by 3) and by 5 (since 165 ends in 5). Thus  $165 = 3 \cdot 5 \cdot 11$ .

**Example:** The divisibility rule for 3 follows from the fact that when a power of 10 is divided by 3 it has remainder 1. For example, 24 is divisible by 3 because

$$24/3 = 2 \cdot \frac{10}{3} + \frac{4}{3} = 2 \cdot \left(3 + \frac{1}{3}\right) + \frac{4}{3} = 2 \cdot 3 + 2 \cdot \frac{1}{3} + \frac{4}{3} = 6 + \frac{2}{3} + \frac{4}{3} = 6 + \frac{2+4}{3}.$$

Therefore,  $\frac{24}{3} = 6 + \frac{2+4}{3}$ , so 24 is divisible by 3 because  $2 + 4$  is divisible by 3.

## 7.3 Understand and work with square and cube roots.

### 7.3a Understand the definition of a *root* of rational number.

- If  $a^n = b$ , so that  $b$  is a *power* of  $a$ , then  $a$  is said to be a *root* of  $b$ .
  - Note:** As subtraction undoes addition and division undoes multiplication, so roots undo powers.
- The most common root is the *square root*, symbolized by  $\sqrt{\quad}$ , corresponding to the second (square) power: If  $a^2 = b$ , then  $a = \sqrt{b}$ .
- If  $a^3 = b$ , then  $b$  is the *cube* of  $a$ , and  $a$  is the *cube root* of  $b$ .
- Know the squares of numbers from 1 to 12 and the cubes of numbers from 1 to 5.

**Note:** Positive integers that are the squares of other integers are called *square numbers*. Knowing the squares of numbers from 1 to 12 carries with it knowledge of the square roots of square numbers between 1 and 144.

### 7.3b Recognize that unless they are integers, square, cube, and $n$ th roots of integers are not rational numbers.

- Such roots are numbers that cannot be written as the ratio of two integers.
  - Note:** This is a consequence of the fact that integers can be uniquely factored into products of prime numbers. The actual proof is a bit subtle.
- Numbers that cannot be written as the ratio of integers are called *irrational*. Square, cube, and  $n$ th roots provide the most common examples of irrational numbers.

### 7.3c Understand why $\sqrt{mn} = \sqrt{m} \cdot \sqrt{n}$ and why $(\sqrt{m})^2 = m$ .

**Note:** This expected property merits special emphasis since it is about irrational numbers and therefore does not follow directly from earlier observations about rational numbers.

**Note:** To verify these properties, we use the definition of root to transform the property to a statement about rational numbers--we already know to be true. Specifically,  $(\sqrt{m} \cdot \sqrt{n})^2 = (\sqrt{m} \cdot \sqrt{n}) (\sqrt{m} \cdot \sqrt{n}) = \sqrt{m} \cdot \sqrt{m} \cdot \sqrt{n} \cdot \sqrt{n} = (\sqrt{m})^2 \cdot (\sqrt{n})^2 = mn$ . Therefore  $\sqrt{mn} = \sqrt{m} \cdot \sqrt{n}$ .

### 7.3d Estimate square and cube roots and use calculators to find good approximations.

- If  $a < n < b$ , then  $\sqrt{a} < \sqrt{n} < \sqrt{b}$ .  
**Example:** Because  $2.6^2 = 6.76$  and  $2.7^2 = 7.29$ ,  $\sqrt{7}$  is between 2.6 and 2.7. That is, since  $6.76 < 7 < 7.29$ , therefore  $2.6 < \sqrt{7} < 2.7$ .
- If an estimate  $a$  to  $\sqrt{n}$  is too high (or too low), then the quotient  $b = n/a$  will be another estimate that is too low (or too high). In such a case, the average  $(a+b)/2$  will be a better next estimate.  
**Note:** This method of successive approximations is used in many mathematical situations where exact answers are hard to calculate.  
**Example:** Since  $7 < 9$ , estimate that  $\sqrt{7} = 2.8$ . Since  $7/2.8 = 2.5$ , this estimate is clearly too high. Next try  $(2.8+2.5)/2 = 2.65$ . Then  $7/2.65 = 2.6415$ . Average once again, and test:  $(2.65+2.6415)/2 = 2.64575$ , and  $7/2.64575 = 2.6457526...$  This gives six significant digits in three repetitions of the process.

7.3e Use rational numbers (including percents) and roots to solve mathematical, contextual, and verbal problems.

## Number & Operations

### 8.1 Understand the definition of irrational numbers and know some examples.

8.1a Name some common examples of irrational numbers and locate them approximately on the number line.

- Numbers that are not rational (i.e., those that cannot be expressed as the ratio of integers) are called *irrational* numbers.
 

**Examples:**  $\sqrt{2}$ ,  $\sqrt{3}$ ,  $\sqrt{5}$ , and  $\pi$  are irrational numbers. (The number  $\pi$  is defined in the Geometry strand.)
- The decimal expansion of an irrational number never ends, and never repeats. (If it did, then the number would be rational.)
- Use the first few digits in the decimal expansion of an irrational number to locate it on the number line.
- Recognize that  $22/7$  and  $3.14$  are just approximations to the irrational number  $\pi$ .

**Note:** When approximations are used in a calculation, the "squiggly" sign  $\approx$  meaning "approximately" must be used rather than the equals sign  $=$ .

**Example:**  $(2 + \sqrt{2})(3 - \sqrt{2}) = 2 \cdot 3 - 2\sqrt{2} + 3\sqrt{2} - 2 = 4 + \sqrt{2} \approx 5.4142$ .

8.1b Use indirect arguments to show that certain numbers are irrational.

- Show that any non-zero rational multiple of an irrational number is irrational.
 

**Example:** Suppose  $(3/8)\sqrt{2}$  were rational. Then for some integers  $m$  and  $n$ ,  $(3/8)\sqrt{2} = m/n$ . That would mean that  $\sqrt{2} = (m/n) \cdot (8/3)$ , which is a rational number. This  $\sqrt{2}$  is actually irrational, this contradiction shows that our original supposition must be incorrect. Hence  $(3/8)\sqrt{2}$  is irrational.
- Show that the square root of a positive integer is either an integer or irrational.

**Example:** Suppose  $\sqrt{2}$  were rational. Then  $\sqrt{2}$  can be expressed as a fraction in lowest terms, i.e.,  $\sqrt{2} = m/n$  for some integers  $m$  and  $n$  where the greatest common divisor of  $m$  and  $n$  is 1. Thus  $m^2 = 2n^2$ . Since  $n$  can be factored uniquely into primes,  $n^2$  must have an even number of prime factors. Hence  $2n^2$ , and thus  $m^2$ , must have an odd number of prime factors. But this cannot be true, since  $m$  also can be factored uniquely into primes, so  $m^2$  must have an even number of prime factors. This contradiction shows that  $\sqrt{2}$  cannot be rational.

**Note:** This classic proof relies on two important properties of integers--the existence of a greatest common divisor, and the unique decomposition into primes.



# MAP Mathematics Expectations

## Data and Measurement

### **K.6 Compare the length, weight, and capacity (volume) of objects.**

- K.6a Make direct comparisons between objects (e.g., recognize which is shorter, longer, taller, lighter, heavier, or holds more).
- K6.b Estimate length, weight, and capacity, and check estimates with actual measurements.
- Select and use appropriate measurement tools (rulers, tape measures, scales, containers, clocks, thermometers).
  - Relate direct comparisons of objects to comparisons of numerical measurements or estimates.

**Example:** Tom, who is four feet tall, is shorter than Jose, who is five feet tall, because 4 is smaller than 5.

### **K.7 Recognize and use words that represent time, temperature, and money.**

- K.7a Recognize and use the words *day, night, morning, afternoon, evening, yesterday, today, tomorrow*.
- Identify daily landmark times such as *bedtime* or *lunch time*.
- K.7b Recognize the role of clocks and calendars in measuring and keeping track of time.
- K.7c Know that thermometers measure temperature, and that *degree* is the word used to name a temperature.
- K.7d Identify U.S. coins by name.

## Data and Measurement

### 1.4 Measure length, weight, capacity, time, and money.

- 1.4a Use rulers, scales, and containers to measure and compare the dimensions, weight, and capacity (volume) of classroom objects.

**Note:** The concepts of addition and order are intrinsic in quantities such as length, weight and volume (capacity). So measuring such quantities provides an independent empirical basis for understanding the properties of numbers that is different from simple counting.

- Round off measurements to whole numbers.
- Recognize the essential role of units in measurement, and understand the difference between *standard* and *non-standard* units.

**Example:** An inch or foot marked on a ruler is a standard unit, whereas a paperclip or a classmate's foot used to measure is a non-standard unit.

- Represent addition by laying rods of different lengths end to end, combining items on a balance, and pouring liquids or sand into different containers.
- Estimate lengths with simple approximations.
- Understand and use comparative words such as *long, longer, longest; short, shorter, shortest; tall, taller, tallest; high, higher, highest*.

- 1.4b Tell time from analog (round) clocks in half-hour intervals.

- Use the expressions "o'clock" and "half past."

- 1.4c Count, speak, write, add, and subtract amounts of money in cents up to \$1 and in dollars up to \$10.

- Know the values of US coins (penny, nickel, dime, quarter, dollar bill).
- Use the symbols \$ and ¢ separately (e.g., \$4, 35¢ instead of \$4.35).
- Use coins to decompose monetary amounts given in cents.

**Example:**  $17¢ = \text{one dime, one nickel, and two pennies} = 10 + 5 + 1 + 1$ ; or  $17¢ = \text{three nickels and two pennies} = 5 + 5 + 5 + 1 + 1$ .

- Understand and solve money problems expressed in a different ways, including *how much more* or *how much less*.

**Note:** Avoid problems that require conversion from cents to dollars or vice versa.

### 1.5 Use picture graphs to pose and solve problems.

- 1.5a Interpret picture graphs in words (orally) and with numbers.

- Answer questions about the meaning of picture graphs.

- 1.5b Create picture graphs of counts and measurements from collected or provided data.

- Represent data both in horizontal and vertical forms.
- Label axes or explain what they represent
- Pose and answer comparison questions based on picture graphs.

## Data and Measurement

### 2.5 Add, subtract, compare, and estimate measurements.

2.5a Estimate, measure, and calculate length in meters, centimeters, yards, feet, and inches.

- Recognize and use standard abbreviations: m, cm, yd, ft, and in, as well as the symbolic notation 3'6".
- Understand and use units appropriate to particular situations.  
**Example:** Standard U.S. school notepaper is sized in inches, not centimeters.
- Add and subtract mixed metric units (e.g., 8m,10cm + 3m,5cm) but defer calculation with mixed English units (e.g., 3ft,1in + 1ft,8in) until third grade.

**Note:** Conversion between systems awaits a later grade.

2.5b Measure the lengths of sides and diagonals of common two-dimensional figures such as triangles, rectangles (including squares) and other polygons.

- Measure to the nearest centimeter or half inch using meter sticks, yardsticks, rulers, and tape measures marked in either metric or English units.  
**Note:** Measure *within* either system without conversion between systems.
- Create and use hand-made rulers by selecting an unconventional unit length (e.g., a hand-width), marking off unit and half-unit lengths.
- Explore a variety of ways to measure perimeter and circumference.

**Examples:** Encircle with a tape measure; measure and sum various pieces; wrap with a string and then measuring the length of the string. Compare answers obtained by different strategies and explain any differences.

**Note:** Comparing the result of a direct measurement (encircling) with that of adding component pieces underscores the importance of accuracy and serves as a prelude to understanding the significance of significant digits.

2.5c Estimate and measure weight and capacity in common English and metric units.

- Recognize, use, and estimate common measures of volume (quarts, liters, cups, gallons) and weight (pound, kilogram).
- Understand and use common expressions such as *half a cup* or *quarter of a pound* that represent fractional parts of standard units of measurement.

2.5d Compare lengths, weights, and capacities of *pairs* of objects.

- Demonstrate that the combined length of the shorter pieces from two pairs of rods is shorter than the combined lengths of the two longer pieces.
- Recognize that same applies to combined pairs of weights or volumes.

**Note:** Even though this relation may seem obvious, it is an important demonstration of the fundamental relation between addition and order, namely, that if  $a \leq b$  and  $c \leq d$ , then  $a+c \leq b+d$ .

### 2.6 Tell, estimate, and calculate with time.

2.6a Tell, write, and use time measurements from analogue (round) clock faces and from digital clocks and translate between the two.

- Round off to the nearest five minutes.
- Understand and use different ways to read time, e.g., "nine fifteen" or "quarter past nine"; "nine fifty" or "ten to ten."
- Understand a.m. and p.m.

- 2.6b Understand the meaning of time as an interval, and be able to estimate the passage of time without clock measurement.
- 2.6c Understand and use comparative phrases such as "in fifteen minutes," "half an hour from now," "ten minutes late."

## **2.7 Count, add, and subtract money.**

- 2.7a Read, write, add, and subtract money up to ten dollars.
  - Handle money accurately and make change for amounts of \$10 or less by counting up.
  - Use the symbols \$ and ¢ properly
  - Recognize and use conventional ("decimal") monetary notation and translate back and forth into \$ and ¢ notation.
  - Add and subtract monetary amounts in both \$ and ¢ and conventional notation.
  - Use a calculator to check monetary calculations, and also to add lists of three or more amounts.
  - Estimate answers to check for reasonableness of hand or calculator methods.

## **2.8 Represent measurements by means of bar graphs.**

- 2.8a Collect data and record them in systematic form.
- 2.8b Select appropriate scales for a graph, and make them explicit in labels.
  - Employ both horizontal and vertical configurations.
  - Recognize an axis with a scale as a representation of the number line.
  - Compare scales on different graphs.
  - Use addition and subtraction as appropriate to translate data (gathered or provided) into measurements required to construct a graph.
- 2.8c Create and solve problems that require interpretation of bar or picture graphs.

## Data and Measurement

### 3.7 Recognize why measurements need units and know how to use common units.

3.7a Understand that all measurements require units and that a quantity accompanied by a unit represents a measurement.

- Know and use the names and approximate magnitudes of common units:
  - For length: kilometer, meter, centimeter; mile, yard, foot, inch.
  - For capacity: liter, milliliter; gallon, quart, pint, cup.
  - For time: year, month, week, day, hour, minute, second.
  - For money: pennies, nickels, dimes, quarters, dollars.

**Note:** Many of these units have been introduced in prior grades; others will be introduced in later grades. Here some are pulled together for reinforcement and systematic use. Each year in grades 2-6 some new measures should be introduced, and previous ones reinforced. Which are done in which grades is of lesser importance.

3.7b Know common within-system equivalences:

- 1 meter = 100 centimeters, 1 yard = 3 feet, 1 foot = 12 inches.
- 1 liter = 1,000 milliliters, 1 gallon = 4 quarts, 1 quart = two pints.
- 1 year = 12 months, 1 week = 7 days, 1 hour = 60 minutes, 1 minute = 60 seconds.
- 1 dollar = 4 quarters = 10 dimes = 100 pennies, 1 quarter = 5 nickels = 25 pennies, 1 dime = 2 nickels = 10 pennies, 1 nickel = 5 pennies.

3.7c Choose reasonable units of measure, estimate common measurements, use appropriate tools to make measurements, and record measurements accurately and systematically.

- Make and record measurements that use mixed units within the same system of measurement (e.g., feet and inches, hours and minutes).

**Note:** Many situations admit various approaches to measurement. Using different means and comparing results is a valuable activity.

- Understand that errors are an intrinsic part of measurement.
- Understand and use time both as an absolute (12:30 pm) and as duration of a time interval (20 minutes).
- Understand and use idiomatic expressions of time (e.g., "10 minutes past 5," "quarter to 12," "one hour and ten minutes").

3.7d Use decimal notation to express, add, and subtract amounts of money.

**Note:** Dealing with money enables students to become accustomed to decimal notation, i.e.,  $\$1.49 + \$0.25 = \$1.74$ .

3.7e Solve problems requiring the addition and subtraction of lengths, weights, capacities, times, and money.

- Include use of common abbreviations: m, cm, kg, g, l, ml, hr, min, sec, in, ft, lb, oz, \$, ¢.

**Note:** Add and subtract only within a single system, using quantities within students' experience. Use real data where possible, but limit the size and complexity of numbers so that problem solving, not computation, is the central challenge of each task.

## Data and Measurement

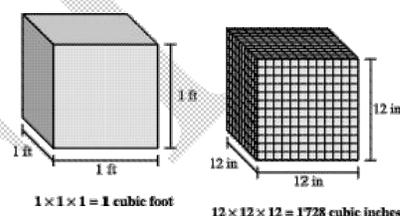
### 4.9 Understand and use standard measures of length, area, and volume.

4.9a Know and use common units of measure of length, area, and volume in both metric and English systems.

- Always use units when recording measurements.
- Know both metric and English units: centimeter, square centimeter, cubic centimeter; meter, square meter, cubic meter; inch, square inch, cubic inch; foot, square foot, cubic foot.
- Use abbreviations: m, cm, in, ft, yd;  $m^2$ ,  $cm^2$ ,  $in^2$ ,  $ft^2$ ,  $yd^2$ ; sq m, sq cm, sq in, sq ft, sq yd;  $m^3$ ,  $cm^3$ ,  $in^3$ ,  $ft^3$ , and  $yd^3$ .

4.9b Convert measurements of length, weight, area, volume, and time within a single system.

**Note:** Emphasize conversions that are common in daily life. Common conversions typically involve adjacent units, for example, hours and minutes or minutes and seconds, but not hours and seconds. Know common within-system equivalences.



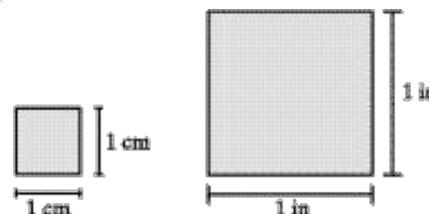
- Use unit cubes to build solids of given dimensions and find their volumes.
- 1 square foot =  $12^2$  square inches; 1 square meter as  $100^2$  square centimeters; 1 cubic foot =  $12^3$  cubic inches; 1 cubic meter as  $100^3$  cubic centimeters.

4.9c Visualize, describe, and draw the relative sizes of length, area, and volume units in the different measurement systems.

- Estimate areas of rectangles in square inches and square centimeters.

**Note:** Avoid between-system conversions.

**Examples:** Centimeter vs. inch, foot and yard vs. meter; square centimeter vs. square inch; square yard vs. square meter; cubic foot vs. cubic meter.



4.9d Recognize that measurements are never exact.

- Both recorded data and answers to calculations should be rounded to a degree of precision that is reasonable in the context of a given problem and the accuracy of the measuring instrument.

**Note:** All measurements of continuous phenomena such as length, capacity or temperature are approximations. Measurements of discrete items such as people or bytes can be either exact (e.g. size of an athletic team) or approximate (e.g., size of a city).

4.9e Solve problems involving area, perimeter, surface area, or volume of rectangular figures.

- Select appropriate units to make measurements on everyday objects, record measurements to reasonable degree of accuracy, and use a calculator when appropriate to compute answers.
- Know that answers to measurement problems require appropriate units in order to have any meaning.

**Note:** Include figures whose dimensions are given as fractions or mixed numbers.

**4.10. Record, arrange, present, and interpret data using tables and various types of graphs.**

4.10a Create and interpret line, bar, and circle graphs and their associated tables of data.

- Create and label appropriate scales for graphs.
- Prepare labels or captions to explain what a table or graph represents.
- Solve problems using data presented in graphs and tables.
- Employ fractions and mixed numbers, as needed, in tables and graphs.

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## Data and Measurement

### 5.9 Make, record, display, and interpret measurements of everyday objects.

- 5.9a Select appropriate units to make measurements, and include units in answers.
- 5.9b Recognize and use measures of weight, information, and temperature.
- For information: bytes, kilobytes (K, or Kb), megabytes (M), gigabytes (G).  
 $1\text{G} = 1000\text{M}$ ,  $1\text{M} = 1000\text{K}$ ,  $1\text{K} = 1000$  bytes.  
**Note:** Literally, the multiplier is  $1024 = 2^{10}$ , but for simplicity in calculation 1000 is generally used instead.
  - For weight: kilogram (kg), gram (g), pound (lb), ounce (oz).  $1\text{ kg} = 1000\text{ g}$ ,  $1\text{ lb} = 16\text{ oz}$ .
  - For temperature: Centigrade and Fahrenheit degrees.  $32^{\circ}\text{ F} = 0^{\circ}\text{ C}$ ;  $212^{\circ}\text{ F} = 100^{\circ}\text{ C}$ .
- 5.9c Record measurements to reasonable degree of accuracy, using fractions and decimals as needed to achieve the desired detail.
- 5.9d When needed, use a calculator to find answers to questions associated with measurements.
- Understand the role of significant digits in signaling the accuracy of measurements and associated calculations.  
**Example:** Report a city's population as 210,000, not as 211,513.
- 5.9e Create graphs and tables to present and communicate data.

### 5.10 Find, interpret, and use the average (mean) of a set of data.

- 5.10a Calculate the average of a set of data that includes whole numbers, fractions, and decimals.
- Note:** Emphasize that *data* is plural and *datum* is singular, the name for a single number in a set of data.
- Infer characteristics of a data set given the mean and other incomplete information.

## Data and Measurement

### 6.6 Understand the meaning of probability and how it is expressed.

6.6a The probability of an event is a number between zero and one that expresses the likelihood of an occurrence.

- The probability of an occurrence is the ratio of the number of actual occurrences to the number of possible occurrences
- Understand different ways of expressing probabilities--as percentages, decimals, or odds.

**Example:** If the probability of rain is .6, the weather forecaster could say that there is a 60% chance of rain, or that the odds of rain are 6:4 ( or 3:2).

- If  $p$  is the probability that an event will occur, then  $1-p$  is the probability that it will not occur.

**Example:** If the probability of rain is 60%, then the probability that it will not rain is  $100\% - 60\% = 40\%$ . (Equivalently,  $1 - .60 = .40$ )

## Data and Measurement

### 7.4 Collect, organize, and analyze both single variable and two-variable data.

- 7.4a Find and interpret the mean (average), median, upper-, lower-, and inner-quartile range of a set of data.
- Prepare and interpret box-and-whisker plots and stem-and-leaf plots.
- 7.4b Interpret and employ various graphs and charts to represent data faithfully.
- Select suitable graph type (e.g., bar graphs, line graphs, circle graphs (pie charts), scatter plots, box-and-whisker plots, and stem-and-leaf plots) and use it to create accurate representations of given data.  
**Note:** All but the last two types have been introduced in earlier grades.
  - Interpret and solve problems using information presented in these various visual forms.
- 7.4c Create and interpret scatter plots.
- Understand the difference between apparent association observed in a scatter plot and legitimate cause and effect.

## Data and Measurement

### 8.2 Understand the relationship between probability and relative frequencies.

8.2a Understand, compute, and graph relative and cumulative frequencies

- Calculate probabilities of events for simple experiments with equally probable outcomes.

**Examples:** Tossing dice, flipping coins, spinning spinners.

- Compare probabilities of two or more events and recognize when certain events are equally likely.

8.2b Solve simple problems involving probability and relative frequency.

- If an action (e.g., throwing a die) is repeated  $n$  times and a certain event (e.g., coming up 5) occurs  $b$  times, the ratio  $b/n$  is called the *relative frequency* of the event occurring.

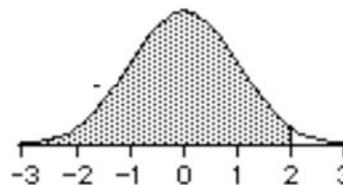
**Note:** With increasing numbers of trials relative frequency is increasingly likely to be a good approximation to the true probability, but even a large number of trials might not produce a relative frequency very close to the true probability. (Relative frequency is often called *empirical* or *experimental probability* in contrast with *theoretical probability*.)

- Recognize common misconceptions associated with dependent and independent event

**Examples:** Lotteries, "hot streaks."

8.2c Interpret main features of the graph of the normal distribution.

- Understand common examples that fit the normal distribution (e.g., height, weight).
- Recognize that approximately 2/3 of the cases fall in the middle third of the range of the graph, and that 95% of the cases fall in the middle two-thirds of the range of the graph.



### 8.3 Understand and use ratios, derived quantities, and indirect measurements.

8.3a Solve data problems using ratios, rates, and percentages.

- Understand simple and compound interest.

8.3b Understand and use ratio and multiplicative quantities.

- Ratio quantities include
  - velocity, measured in units such as miles per hour (mph);
  - density, measured in units such as kilograms per liter (kg/l);
  - pressure, measured in units such as pounds per square foot (lb/ft<sup>2</sup>);
  - population density, measured in units such as persons per square mile.
- Multiplicative quantities include
  - area, measured in units such as square feet (ft<sup>2</sup>);
  - volume, measured in units such as cubic meters (m<sup>3</sup>);
  - energy, measured in units such as kilowatt hours (kw-h);
  - work, measured in units such as person-days.
- Make within-system conversions of derived quantities.

**Examples:** Feet per second to miles per hour; square feet to square inches.

**8.3c Plan and carry out direct and indirect measurements.**

- Use similarity and shadows to make indirect measurements.
- Understand how the precision of measurement influences accuracy of quantities derived from these measurements.

**Note:** Analysis of accuracy assumes that measurement errors are small relative to the quantity being measured, as they should be unless an error arises in transcription or data entry.

**Example:** When calculating an area  $A$  by means of the product  $bh$ , if the measurement uncertainty for both  $b$  and  $h$  is  $\pm x\%$ , then the uncertainty in  $A$  is at most  $\pm 2x\%$ . For cubes, an  $x\%$  error in linear measurements yields about  $\pm 3x\%$  error in the volume calculation.

**8.3d Understand the behavior of derived measurements such as weighted averages and percentage change.**

- Recognize instances of weighted averages such as the average grade of students in a school district, the average salary of NBA players, and the consumer price index (CPI).

**8.3e Use ratios to create and interpret scale drawings as a tool for solving problems.**



## MAP Mathematics Expectations

### Geometry

#### K.8 Create, explore and describe shapes.

K.8a Identify common shapes such as rectangle, circle, triangle, and square.

- Draw a variety of triangles (equilateral, right, isosceles, scalene) in a variety of positions.
- Draw squares and rectangles of different proportions (tall, squat, square-like) both horizontally and vertically positioned. Recognize tipped squares and rectangles.

**Note:** Squares are a type of rectangle.

**Note:** *Drawing* of tipped rectangles is typically too hard for kindergarten.

- Describe attributes of common shapes (e.g., number of sides and corners).

K.8b Use geometric tiles and blocks to assemble compound shapes.

- Assemble rectangles from two congruent right triangular tiles.
- Explore two-dimensional symmetry using matching tiles.

K.8c Recognize and use words that describe spatial relationships such as *above, below, inside, outside, touching, next to, far apart.*

## Geometry

### 1.6 Recognize, describe, and draw geometric figures.

1.6a Identify and draw two-dimensional figures.

- Include trapezoids, equilateral triangles, isosceles triangle, parallelograms, quadrilaterals.

**Note:** Be sure to include a robust variety of triangles as examples, especially ones that are very clearly not equilateral or isosceles.

- Describe attributes of two-dimensional shapes (e.g., number of sides and corners).

1.6b Identify and name three dimensional figures.

- Include spheres, cones, prisms, pyramids, cubes, rectangular solids.
- Identify two-dimensional shapes as faces of three-dimensional figures.

1.6c Sort geometric objects by shape and size.

- Recognize the attributes that determined a particular sorting of objects and use them to extend the sorting.

**Example:** Various L-shaped figures constructed from cubes are sorted by the total number of cubes in each. Recognize this pattern, then sort additional figures to extend the pattern.

- Explore simultaneous independent attributes.

**Example:** Sort triangular tiles according to the four combinations of two attributes such as right angle and equal sides.

### 1.7 Rotate, invert, and combine geometric tiles and solids.

1.7a Describe and draw shapes resulting from rotation and flips of simple two-dimensional figures.

- Identify the same (congruent) two-dimensional shapes in various orientations and move one on top of the other to show that they are indeed identical.
- Extend sequences that show rotations of simple shapes.

1.7b Identify symmetrical shapes created by rotation and reflection.

1.7c Use geometric tiles and cubes to assemble and disassemble compound figures.

- Count characteristic attributes (lines, faces, edges) before and after assembly.

**Examples:** Add two right triangles to a trapezoid to make a rectangle; create a hexagon from six equilateral triangles; combine two pyramids to make a cube.

## Geometry

### 2.9 Recognize, classify, and transform geometric figures in two and three dimensions.

2.9a Identify, describe, and compare common geometric shapes in two and three dimensions.

- Define a general triangle and identify isosceles, equilateral, right, and obtuse special cases.

**Note:** The goal of naming triangles is not the names themselves but to focus on important differences. Triangles (or quadrilaterals) are not all alike, and it is their differences that give them distinctive mathematical features.

- Identify various quadrilaterals (rectangles, trapezoids, parallelograms, squares) as well as pentagons and hexagons.

**Note:** In this grade *parallel* is used informally and intuitively; it receives more careful treatment at a later grade.

**Note:** A *square* is a special kind of rectangle (since it has four sides and four right angles); a *rectangle* is a special kind of parallelogram (since it has four sides and two pairs of parallel sides); and a *parallelogram* is a special kind of *trapezoid* (since it has four sides and at least one pair of parallel sides); and a trapezoid is a polygon (since it is a figure formed of several straight sides). So for example, contrary to informal usage, in mathematics a square is a trapezoid.

- Understand the terms *perimeter* and *circumference*.

**Note:** The primary meaning of both terms is the outer boundary of a two-dimensional figure; circumference is used principally in reference to circles. A secondary meaning for both is the length of the outer boundary. Which meaning is intended needs to be determined from context.

- Distinguish circles from ovals; recognize the circumference, diameter, and radius of a circle.
- In three dimensions, identify spheres, cones, cylinders, triangular and rectangular prisms.

2.9b Describe common geometric attributes of familiar plane and solid objects.

- Common geometric attributes include position, shape, size, roundness, and numbers of corners, edges, and faces.
- Distinguish between geometric attributes and other characteristics such as weight, color, or construction material.
- Distinguish between lines and curves, and between flat and curved surfaces.

2.9c Rotate, flip, and fold shapes to explore the effect of transformations.

- Use paper folding to find lines of symmetry.
- Recognize congruent shapes.
- Identify shapes that have been moved (flipped, slid, rotated), enlarged, or reduced.

### 2.10 Understand and interpret rectangular arrays as a model of multiplication.

2.10a Create square cells from segments of the discrete number line used as sides of a rectangle.

- Match cells to discrete objects lined up in regular rows of the same length.

2.10b Understand rectangular arrays as instances of repeated addition.

## Geometry

### 3.8 Recognize basic elements of geometric figures and use them to describe shapes.

- 3.8a Identify points, rays, line segments, lines, planes in both mathematical and everyday settings.



- A *line* is a straight path traced by a moving point having no breadth nor end in either direction.  
**Examples:** Each figure on the left above represents a line; the arrows indicate that the lines keep going in the indicated directions without end. The number line with both positive and negative numbers is a line.
- A part of a line that starts at one point and ends at another is called a *line segment*. Line segments are drawn without arrows on either end because line segments end at points.  
**Examples:** The figure in the center above is a line segment. The edges of a desk or door or piece of paper are everyday examples of line segments.
- Part of a line that starts at one point and goes on forever in one direction is called a *ray*.  
**Examples:** The figure above on the right is a ray. The positive number line (to the right of 0) is a ray. On the other hand, none of the four examples at the right are lines:  
**Caution:** Not all sources distinguish carefully among the terms *line*, *segment*, or *ray*, nor do all sources employ the convention of arrowheads in exactly the manner described above. Often context is the best guide to distinguish among these terms.
- Know that a *plane* is a flat surface without thickness that extends indefinitely in every direction.  
**Examples:** Everyday examples that illustrate a part of a plane are the flat surfaces of a floor, desk, windowpane, or book. Examples that are not part of a plane are the curved surfaces of a light bulb, a ball, or a tree.



- 3.8b Understand the meaning of parallel and perpendicular and use these terms to describe geometric figures.

- Lines and planes are called *parallel* if they do not meet no matter how far they are extended
- Lines and planes are called *perpendicular* if the corners formed when they meet are equal.
- Identify parallel and perpendicular edges and surfaces in everyday settings (e.g., the classroom).  
**Examples:** The lines on the page of a notebook are parallel, as are the covers of a closed book are parallel. Corners of books, walls, and rectangular desks are perpendicular, as are the top and side edges of a chalk board and a wall and a floor in a classroom.
- The corner where two perpendicular lines meet is called a *right angle*.  
**Note:** The general concept of "angle" is developed later; here the term is used merely as the name for this specific and common configuration.
- Understand and use the terms *vertical* and *horizontal*.

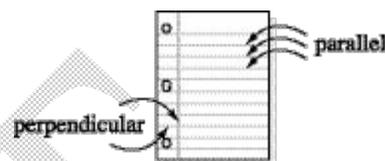
- Recognize that vertical and horizontal lines or planes are perpendicular, but that perpendicular lines or planes are not necessarily vertical or horizontal.

3.8c Use terms such as line, plane, ray, line segment, parallel, perpendicular, and right angle correctly to describe everyday and geometric figures.

### 3.9 Identify and draw perpendicular and parallel lines and planes.

3.9a Draw perpendicular, parallel, and non-parallel line segments using rulers and squares.

- Recognize that lines that are parallel to perpendicular lines will themselves be perpendicular.



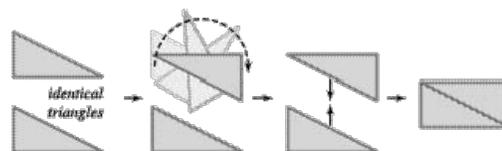
**Example:** Fold a piece of paper in half from top to bottom, then fold it in half again from left to right. This will give two perpendicular fold lines and four right angles.

- Edges of a polygon are called parallel or perpendicular if they lie on parallel or perpendicular lines, respectively.
- Similarly, faces of a three-dimensional solid are called parallel or perpendicular if they lie in parallel or perpendicular planes, respectively.

### 3.10 Explore and identify familiar two- and three-dimensional shapes.

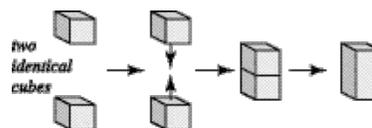
3.10a Describe and classify plane figures and solid shapes according to the number and shape of faces, edges, and vertices.

- Plane figures include circles, triangles, squares, rectangles, other polygons); solid shapes include spheres, pyramids, cubes, and rectangular prisms.
- Recognize that the exact meaning of many geometric terms (e.g., rectangle, square, circle, and triangle) depends on context: sometimes they refer to the boundary of a region and sometimes to the region contained within the boundary.



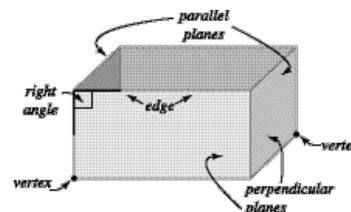
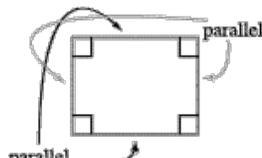
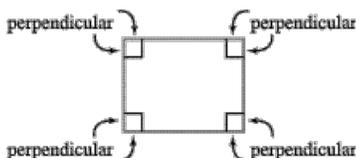
3.10b Know how to put shapes together and take them apart to form other shapes.

**Examples:** Two identical right triangles can be arranged to form a rectangle (see figure above). Two identical cubes can be arranged to form a rectangular prism (figure at right).



3.10c Identify edges, vertices (corners), perpendicular and parallel edges, and right angles in two-dimensional shapes.

**Example:** A rectangle has four pairs of perpendicular edges, two pairs of parallel edges, and four right angles.

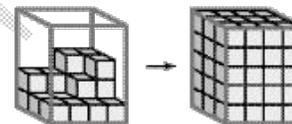


3.10d Identify right angles, edges, vertices, perpendicular and parallel planes in three-dimensional shapes.

### 3.11 Understand how to measure length, area, and volume.

3.11a Understand that measurements of length, area, and volume are based on standard units.

- Fundamental units are: a *unit interval* of length 1 unit, a *unit square* whose sides have length 1 unit, and a *unit cube* whose sides have length 1 unit.
- The *volume* of a rectangular prism is the number of unit cubes required to fill it exactly (with no space left over).



**Note:** The common childhood experience of pouring water or sand offers a direct representation of volume.

- The *area* of a rectangle is the number of unit squares required to pave the rectangle--that is, to cover completely without any overlapping.
- **Note:** Area provides a critical venue for developing the conceptual underpinnings of multiplication.
- The *length* of a line segment is the number of unit intervals that are required to cover the segment exactly with nothing left over.

3.11b Know how to calculate the perimeter, area, and volume of shapes made from rectangles and rectangular prisms.

- The *perimeter* of a rectangle is the number of unit intervals that are required to enclose the rectangle.
- Measure and compare the areas of shapes using non-standard units (e.g., pieces in a set of pattern blocks).
- Recognize that the area of a rectangle is the product of the lengths of its base and height ( $A = b \times h$ ), and that the volume of a rectangular prism is the product of the lengths of its base, width, and height ( $V = b \times w \times h$ ).

**Example:** Build solids with unit cubes and use the formula for volume ( $V=bwh$ ) to verify the count of unit cubes; make similar comparisons with rectilinear figures in the plane that are created from unit squares.

- Find the area of a complex figure by adding and subtracting areas.
- Compare rectangles of equal area and different perimeter, and also rectangles of equal perimeter and different area.
- Measure surface area of solids by covering each face with copies of a unit square, and then counting the total number of units.

## Geometry

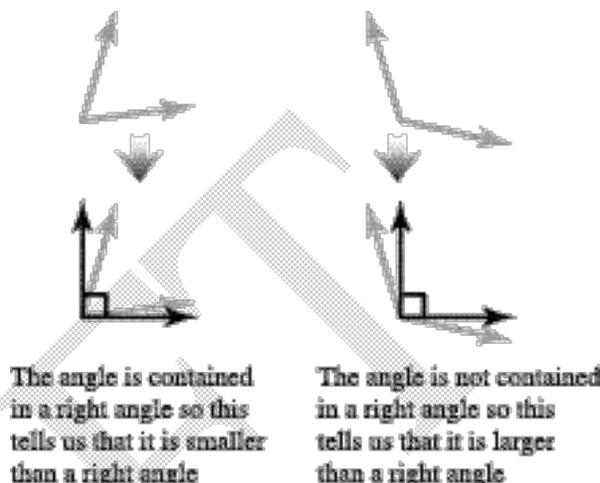
### 4.11 Understand and use the definitions of angle, polygon and circle.

4.11a An angle in a plane is a region between two rays that have a common starting point.

**Note:** According to this definition, a right angle (as determined by perpendicular rays) is indeed an angle.

4.11b If angle A is contained in another angle B, then angle B is said to be bigger than angle A.

- The figure on the right illustrates how to determine whether an angle is larger than, smaller than, or close to a right angle.

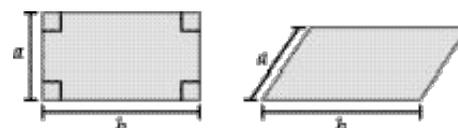


**Note:** When two rays come from the same point (see figure at right) they divide the plane into two regions, giving two angles. Except where otherwise indicated, the angle is determined by the two rays is defined, by convention, as the smaller region.



- Understand that shapes such as triangles, squares, rectangles have angles.  
**Note:** Technically, polygons do not contain rays, which are required for the definition of angles. Their sides are line segments of finite length. Nonetheless, if we imagine the sides extending indefinitely away from each corner, then each corner becomes an angle.

**Example:** Describe the difference between the two figures on the right:



- Identify *acute*, *obtuse*, and *right* angles.

4.11c Know and use the basic properties of squares, rectangles, and isosceles, equilateral, and right triangles.

- Identify *scalene*, *acute* and *obtuse* triangles.
- Know how to mark squares, rectangles, and triangles appropriately:



4.11d Know what a polygon is and be able to identify and draw some examples.

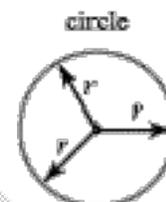
- A *polygon* is a figure that lies in a plane consisting of a finite number of line segments called *edges* (or *sides*) with the properties that (a) each edge is joined to exactly two other edges at the end points; edges do not meet each other except at end points; and the edges enclose a single region.

**Example:** The figures on the right are *not* polygons:



4.11e Know and use the basic properties of a circle.

- A circle as the set of points in a plane that are at a fixed distance from a given point.
- Know that a circle is not a polygon.



## Geometry

### 5.11 Measure angles in degrees and solve related problems.

5.11a Understand the definition of *degree* and be able to measure angles in degrees.

- A *degree* is one part of the circumference of a circle of radius 1 unit (a *unit circle*) that is divided into 360 equal parts. The measure of an angle in degrees is defined to be the number of degrees of the arc of the unit circle, centered at the vertex of the angle, that is intercepted by the angle.
- The measure of an angle in degrees can also be interpreted as the amount of counter-clockwise turning from one ray to the other.

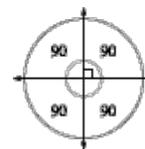


**Note:** Earlier (in Grade 4) the angle determined by two rays was defined to be *smaller* of the two options. For consistency, therefore, when an angle is measured by the amount of turning necessary to rotate one ray into another, it is important to start with the particular ray that will produce an angle measure no greater than  $180^\circ$

- The symbol  $^\circ$  is an abbreviation for "degree" (e.g., 45 degrees =  $45^\circ$ ).
- As a shorthand, angles are called equal if the measures of the angles are equal.

5.11b Know and use the measures of common angles.

- Recognize that angles on a straight line add up to  $180^\circ$  and that angles around a point add up to  $360^\circ$ . An angle of  $180^\circ$  is called a *straight angle*.
- A right angle is an angle of  $90^\circ$ . An acute angle is an angle of less than  $90^\circ$ , while an obtuse angle is an angle of more than  $90^\circ$



**Note:** Since a pair of perpendicular lines divides the plane into 4 equal angles, the measure of a right angle is  $360^\circ/4 = 90^\circ$ .

5.11c Interpret and prepare circle graphs (pie charts).

### 5.12 Know how to do basic constructions using a straight edge and compass.

5.12a Basic constructions include (a) drop a perpendicular from a point to a line, (b) bisect an angle; (c) erect the perpendicular bisector of a line, and (d) construct a hexagon on a circle.

**Note:** A straightedge is a physical representation of a line, not a ruler which is used for measuring. The role of a straightedge in constructions is to draw lines through two points, just as the role of the compass is to draw a circle based on two points, the center and a point on the circumference.

**Note:** Students need extended practice with constructions since they embody the elements of geometry--lines and circles--independent of numbers and measurement. Since constructions are so central to Euclidean geometry, they are often called *Euclidean constructions*.

**Note:** These constructions are *basic* in the sense that other important constructions introduced in later grades (e.g., of an equilateral triangle given one side; of a square inscribed in a circle) build on them.

- Use informal arguments such as paper folding to verify the correctness of constructions.

### 5.13 Recognize and work with simple polyhedra.

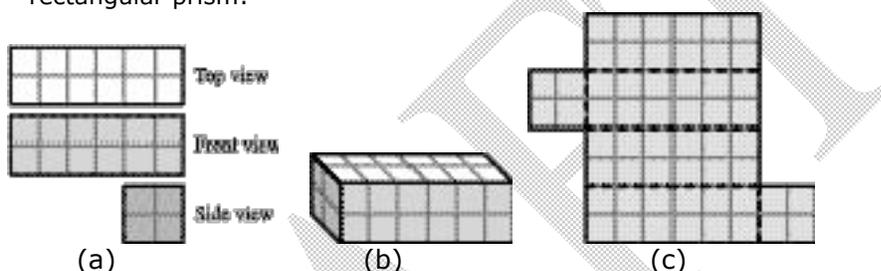
5.13a Represent and work with rectangular prisms by means of orthogonal views, projective views, and nets.

- A *net* is a flat (two-dimensional) pattern of faces nets that can be folded to form the surface of a solid.

**Note:** Because a net represents the surface of a polyhedra spread out in two dimensions, the area of a net equals the surface area of the corresponding solid.

- Orthogonal views are from top, front and side; picture views are either projective or isometric; and nets are plane figures that can be folded to form the surface of the solid.

**Example:** An orthogonal view (a), a projective view (b) and a net (c) of the same rectangular prism:



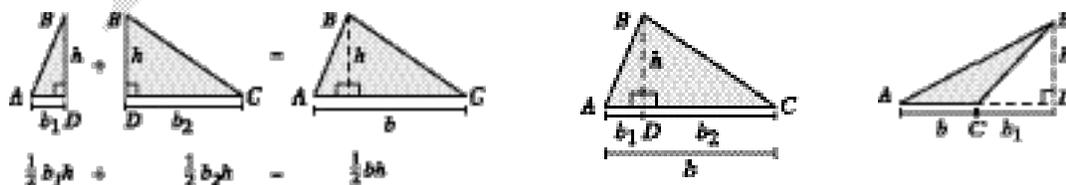
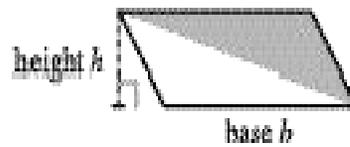
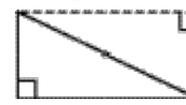
5.13b Recognize the five regular ("Platonic") solids.

- Count faces, edges, and vertices, and make a table with the results.

### 5.14 Find the area of shapes created out of triangles.

5.14a Understand, derive, and use the formula  $A = \frac{1}{2}bh$  for the area of a triangle.

- Arrange two identical right triangles with base  $b$  and height  $h$  to form a rectangle whose area is  $bh$ . Since the area of each right triangle is half that of the rectangle,  $A = \frac{1}{2}bh$ .
- If triangle ABC is not a right triangle, then placing two copies together will form a parallelogram with base  $b$  and height  $h$ . This parallelogram can be transformed into a rectangle of area  $bh$  by moving a right triangle of height  $h$  from one side of the parallelogram to the other. So here too,  $A = \frac{1}{2}bh$ .
- Alternatively, to divide a general triangle ABC into two right triangles as shown below, and combine the areas of the two parts:



**Note:** As the diagrams show, there are two cases to consider: For an acute triangle (where all angles are smaller than a right angle), the parts are added together. For an obtuse triangle (where one angle is larger than a right angle), one right triangle must be subtracted from the other.

5.14b Find the area of a convex polygon by decomposing it into triangles.

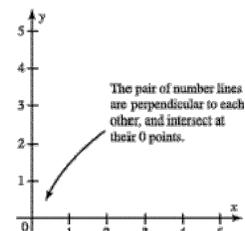
- A polygon is called *convex* if a line segment joining any two points on the perimeter of the polygon will lie inside or on the polygon.
- Any convex polygon of  $n$  sides can be decomposed into  $(n-2)$  triangles.

5.14c Find the area of other geometric figures that can be paved by triangles.

### 5.15 Interpret and plot points on the coordinate plane.

5.15a Associate an ordered pair of numbers with a point in the first (upper right) quadrant and, conversely, any such point with an ordered pair of numbers.

- Positions on the coordinate plane are determined in relation to the coordinate axes, a pair of number lines that are placed perpendicular to each other so that the zero point of each coincides.



**Note:** The coordinate plane is a two-dimensional extension of the number line and builds on extensive (but separate) prior work with the number line and with perpendicular lines.

- Recognize the similarity between locating points on the coordinate plane and locating positions on a map.
- Recognize and use the terms *vertical* and *horizontal*.

5.15b Identify characteristics of the set of points that define vertical and horizontal line segments.

- Use subtraction of whole numbers, fractions, and decimals to find the length of vertical or horizontal line segments identified by the ordered pairs of its endpoints.

**Example:** What is the length of the line segment determined by  $(3/5, 0)$  and  $(1.5, 0)$ ?

## Geometry

### 6.7 Understand and use basic properties of triangles and quadrilaterals.

6.7a Understand and use the angle properties of triangles and quadrilaterals.

- The sum of angles in a right triangle is  $180^\circ$  since two identical right triangles form a rectangle,



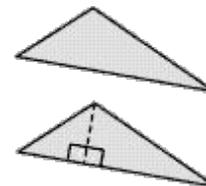
**Note:** By definition a rectangle has 4 right angles, so the sum of the angles of a rectangle is  $4 \times 90^\circ = 360^\circ$ . Each right triangle contains half  $360^\circ$ , or  $180^\circ$ .

**Note:** Since one angle in a right triangle is  $90^\circ$ , the sum of the remaining two angles is also  $90^\circ$ .

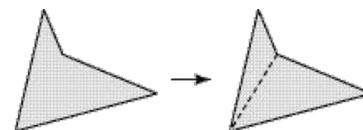
- Since the sum of angles in a parallelogram is also  $360^\circ$ , the sum of angles in any triangle is also  $360^\circ/2 = 180^\circ$ .

**Note:** Following the line of argument used in Grade 5 to find the area of a triangle, we note that (a) two identical copies of any triangle can be arranged to form a parallelogram, and (b) any parallelogram, can be transformed into a rectangle with the same angle sum by moving a triangle from one side of the parallelogram to the other.

**Note:** Alternatively, following the secondary argument offered in Grade 5, one can drop a perpendicular to divide any triangle into two right triangles. The sum of the interior angles of each of these right triangles is  $180^\circ$ , but when put together they include two superfluous right angles. Subtracting these yields  $180^\circ + 180^\circ - (90^\circ + 90^\circ) = 180^\circ$  as the sum of the interior angles of any triangle.



- Since any quadrilateral can be divided into two triangles the sum of the angles in a quadrilateral is also  $2 \times 180^\circ = 360^\circ$ .



6.7b Use a protractor, ruler, square, and compass to draw triangles and quadrilaterals from data given in either numerical or geometric form.

- Draw a variety of triangles (right, isosceles, acute, obtuse) and quadrilaterals (squares, rectangles, parallelograms, and trapezoids) of different dimensions.
- Verify basic properties of triangles and quadrilaterals by direct measurement.

**Note:** Verification by measurement requires many examples, especially some with relatively extreme or uncommon dimensions.

**Examples:** In parallelograms, opposite sides and opposite angles are equal; in rectangles, diagonals are equal.

**Example:** Cut any triangle out of paper and tear it into three parts so that each part contains one of the triangle's vertices. Notice that when the angles are placed together, the edge is straight ( $180^\circ$ ).



- Explore properties of triangles and quadrilaterals with dynamic geometry software.

**Note:** Is a parallelogram a trapezoid? It depends on the definition of trapezoid. If a trapezoid is defined as a quadrilateral with *at least* one pair of parallel edges, parallelograms become special cases of trapezoids. However, dictionaries usually define a trapezoid as a quadrilateral with *exactly one* pair of parallel edges, thereby distinguishing between parallelograms and trapezoids. Mathematicians generally prefer nested definitions as conditions become more or less restrictive. For example, all positive whole numbers are integers, all integers are rational numbers, and all rational numbers are real numbers. So in the world of mathematics, squares are rectangles, rectangles are parallelograms, and parallelograms are trapezoids.

## 6.8 Understand and use basic properties of angles, lines and triangles in the plane.

6.8a Understand the triangle inequality, verify it through measurement, and recognize when it can be useful to solve problems.

- In words, the longest side of a triangle is shorter than the sum of the other two sides. In symbols, if  $a$ ,  $b$ , and  $c$  are three sides of a triangle with  $a \leq b \leq c$ , then  $c < a + b$ .

**Note:** Since those sides of a triangle that are not the longest are by definition shorter than the longest, they too are obviously shorter than the sum of the other two. Thus the triangle inequality applies to all three sides:  $a < b + c$ ,  $b < a + c$ , and  $c < a + b$ .

**Note:** Although relatively simple, the triangle inequality is a deep and fundamental insight that recurs throughout advanced mathematics.

6.8b Know the definitions and properties of interior and exterior angles.

- Understand why each exterior angle of a triangle is equal to the sum of the opposite interior angles.

**Note:** In the triangle shown,  $a + b + c = 180^\circ$  and also  $d + c = 180^\circ$  (since  $c$  and  $d$  are supplementary angles. Therefore  $d = a + b$ .



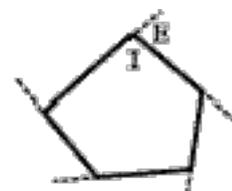
6.8c Understand why the sum of the interior angles of an  $n$ -sided convex polygon is  $(n - 2) \times 180^\circ$ .

- Strategy: decompose an  $n$ -sided polygon into  $n - 2$  triangles.

6.8d Understand why the sum of exterior angles of a convex polygon is  $360^\circ$ .

- A person walking around the perimeter would make one complete revolution. So the sum of exterior angles must be  $360^\circ$ .

**Note:** Here is a more formal explanation for a pentagon. The sum of each adjacent interior angle (I) and exterior angle (E) is  $180^\circ$ . Since there are 5 such pairs of angles, the sum of all interior and exterior angles of a pentagon is  $5 \times 180^\circ = 900^\circ$ . As noted above, the sum of the interior angles is  $(5-2) \times 180^\circ = 540^\circ$ . Subtracting, we get  $900^\circ - 540^\circ = 360^\circ$  as the sum of the exterior angles.

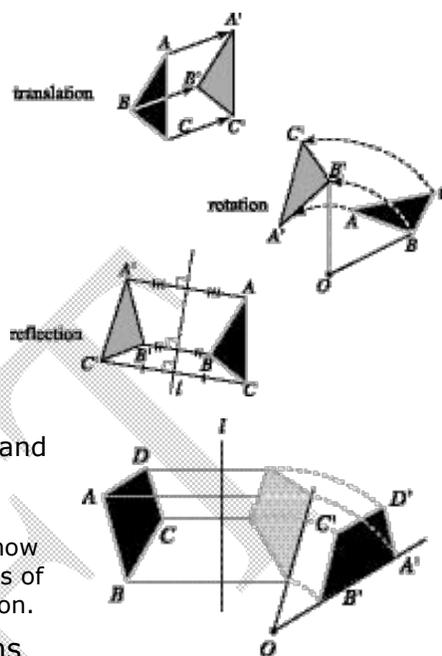


## 6.9 Understand and use the concepts of translation, rotation, reflection, and congruence in the plane.

6.9a Recognize that every rigid motion of a polygon in the plane can be created by some combination of translation, rotation, and reflection.

- Translation, rotation, and reflection move a polygonal figure in the plane from one position to another without changing its length or angle measurements.
- Explore the meaning of rotation, translation, and reflection through drawings and hands-on experiments.

**Example:** The figure on the right illustrates how a rigid motion can be decomposed into a series of three steps: translation, reflection, and rotation.



6.9b Understand several different characterizations and examples of congruence.

- Two figures in the plane are called *congruent* if they have the same size and same shape.
- Two shapes are congruent if they can be made to coincide when superimposed by means of a rigid motion.
- Two polygons are congruent if they have the same number of sides and if their corresponding sides and angles are equal.

**Note:** Historically--beginning with Euclid--congruence applied only to polygons, and used this as the definition. Indeed, the important properties of congruence are typically only about polygonal figures.

- Congruent figures in the plane are those that can be laid on top of one another by rotations, reflections, and translations.

**Note:** Using rotations, reflections, and translations to define congruence gives precise meaning to the intuitive idea of congruence as "same size and same shape," thus permitting a precise definition of congruence for shapes other than polygons.

**Note:** Technological aids (transparencies, dynamic geometry programs) help greatly in studying rigid motions.

6.9c Identify congruent polygonal figures.

- Understand why the two triangles formed by drawing a diagonal of a parallelogram are congruent.
- Understand why the two triangles formed by bisecting the vertex angle of an isosceles triangle are congruent.

## 6.10 Understand and use different kinds of symmetry in the plane.

6.10a Symmetries in the plane are actions that leave figures unchanged.

- Explore and explain the symmetry of geometric figures from the standpoint of rotations, reflections and translations.

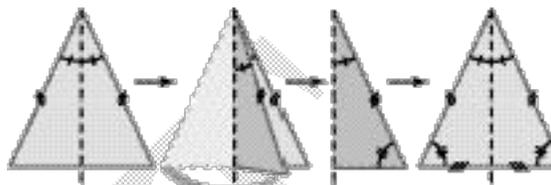
6.10b Identify and utilize bilateral and rotational symmetry in regular polygons.

- A *regular polygon* is a polygon whose sides and angles are all equal.
- *Bilateral symmetry* means there is a reflection that leaves everything unchanged.
- *Regular polygons* have rotational symmetries.

6.10c Identify and utilize translational symmetry in tessellations of the plane.

- Most tessellations have translation symmetry.

6.10d Use reflections to study isosceles triangles and isosceles trapezoids.



- Understand why the base angles of an isosceles triangle are equal
- Understand why the bisector of the angle opposite the base is the perpendicular bisector of the base.

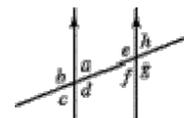
**Note:** Draw an angle bisector on an isosceles triangle. Fold the drawing along the angle bisector (that is, reflect across the angle bisector). Then the base vertices collapse on each other: both angles are equal, thus the angle bisector also bisects the base.

## Geometry

### 7.5 Understand angle properties of parallel lines in the plane.

7.5a Understand the definitions and properties of vertical, adjacent, complementary, supplementary, corresponding, and alternate interior angles.

- When a line intersects two parallel lines, angles  $a$  and  $c$  are *vertical*,  $a$  and  $b$  are *supplementary*,  $a$  and  $h$  are *corresponding* and  $a$  and  $f$  *alternate interior angles*.



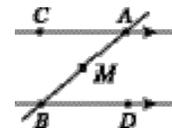
7.5b Understand why vertical angles are equal.

- Vertical angles are equal because a straight angle is  $180^\circ$  and each vertical angle is the supplement of a single angle. In the diagram above,  $a + b = 180^\circ = b + c$ , so  $a = c$ .

7.5c Recognize that corresponding and alternate interior angles are equal.

- (a) If a line intersects two parallel lines, corresponding angles and alternate interior angles must be equal.

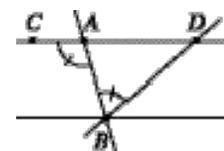
**Note:** To see that alternate interior angles are equal, consider the diagram at the right where lines  $CA$  and  $BD$  are parallel. Let  $M$  be the midpoint of line segment  $AB$ . Now rotate the line  $BD$   $180^\circ$  around the point  $M$ . Then the point  $B$  falls on point  $A$ , and the ray  $BD$  becomes the ray  $AC$  because the lines are parallel. Therefore  $\angle MBD$  falls on  $\angle MAC$ , which means that the two angles are equal.



**Note:** In asserting that ray  $BD$  becomes ray  $AC$ , the preceding argument assumes that there can be no more than one line through point  $A$  that is parallel to the line through  $BD$ . This makes hidden use of Euclid's famous fifth postulate about parallel lines.

- (b) If a line intersects two other lines in such a way as to make the corresponding angles and alternate interior angles equal, then these two lines must be parallel.

**Note:** To show this, suppose two lines make equal alternate interior angles,  $\angle CAB$  and  $\angle ABD$  as shown. Suppose that lines  $CA$  and  $BD$  were not parallel. Then the two lines must intersect at some point, say  $D$  (see diagram). Then points  $ABD$  are the vertices of a triangle.



As shown in Grade 6, the exterior angle  $\angle CAB$  of this triangle equals the total of the two interior angles:  $\angle CAB = \angle ABD + \angle ADB$ .

However, this contradicts the assumption that  $\angle CAB = \angle ABD$ . So our supposition that  $CA$  and  $BD$  were not parallel must not be true. In other words,  $AC$  and line  $BD$  are parallel.

**Note:** The second assertion is the *converse* of the first. Whereas (a) says that  $P$  implies  $Q$ , (b) says that  $Q$  implies  $P$ . Together they form an "if and only if" relationship: "If a line  $L$  intersects two other lines, corresponding and alternate interior angles are equal if and only if these two other lines are parallel."

7.5d Recognize the meanings and roles of assumption, conclusion, converse, and indirect argument.

- Propositions in geometry take the form "if  $P$ , then  $Q$ ."  $P$  is the *assumption* (or *hypothesis*),  $Q$  is the *conclusion*.
- The proposition "if  $Q$ , then  $P$ " is the *converse* of "if  $P$ , then  $Q$ ."

**Note:** Sometimes the converse of a proposition is true, but usually it is not.

**Examples:** Statements (a) and (b) below are converses of each other, as are statements (c) and (d). Statements (a) and (b) are both true, as is statement (c). However, statement (d) is false.

- (a) If a parallelogram is a rectangle, then its diagonals are equal.
- (b) If the diagonals of a parallelogram are equal then the parallelogram is a rectangle.
- (c) If a quadrilateral is a rectangle, then its diagonals are equal.
- (d) If the diagonals of a quadrilateral are equal, then the quadrilateral is a rectangle.

- An *indirect argument* is a method of reasoning that shows that a conclusion cannot be false rather than showing directly that it is true.

**Note:** The proof given above that if alternate interior angles are equal, the lines creating them be parallel is an indirect argument. In particular, it is an *argument by contradiction*. Instead of showing directly that the assumption (equal alternate angles) implies the conclusion (parallel lines), it shows that if the lines are *not* parallel then the two angles would *not* be equal. Since we know that they are equal (by assumption), the lines cannot *not* be parallel. So they must be parallel. There is no other alternative. Arguments by contradiction are very common in mathematics, and that is one reason that this particular result is worth emphasizing.

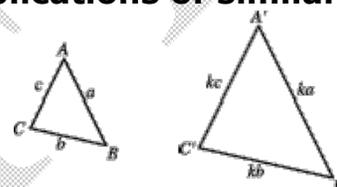
## 7.6 Understand the definition, criteria, and applications of similar triangles.

7.6a Understand the definition of similarity for triangles.

- Informally, two triangles are *similar* if they have the same shape.
- Formally, two triangles are similar if their corresponding angles are equal.

**Note:** The common meaning of *similar* is having the same shape. It is easy to illustrate using the magnification feature of copiers and computers that for triangles, having the same angles yields the same shape.

**Caution:** This is not true for quadrilaterals. All rectangles have equal corresponding angles (they are all  $90^\circ$ ) but they are not all the same shape.



7.6b Recognize several criteria for similarity of triangles.

- Two triangles are similar if the ratios of the lengths of corresponding sides are equal (SSS criterion).

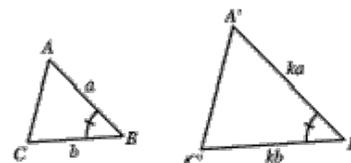
**Note:** Verification should be empirical, using rulers and protractors.

- Two triangles are similar if the ratios of the lengths of two pairs of corresponding sides are equal and the corresponding angles between them are equal (SAS criterion).

**Example:** If  $AB/A'B' = BC/B'C'$  and  $\angle B = \angle B'$ , then triangle ABC is similar to triangle A'B'C'.

- Triangles are similar if two pairs of corresponding angles are equal (AA criterion).

**Note:** The converses of all three of these criteria are all true: if two triangles are similar, they have the properties named in the criteria. Thus they could as well be stated as "if and only if" criteria.



7.6c Recognize that when a line is drawn inside a triangle parallel to one side, it forms a smaller triangle similar to the original one.

- Use equality of ratios of different line segments formed when a line inside a triangle is drawn parallel to one side.
- If points D and E in triangle ABC are on sides AB and AC, respectively, and

the line DE is parallel to BC, then  $AD:AM = AE:AC$  and  $AD:DB = AE:EC$ .

**Note:** The notation  $AD:AM$  stands for the ratio of the lengths of line segment AD to that of line segment AM. The expression  $AD:AM = AE:AC$  symbolizes the "equality of ratios." If AD is three times as long as AM, then AE will be three times as long as EC.

7.6d Use similar triangles to find the lengths of unknown line segments in a triangle.

- Employ the notion of similarity to solve geometric problems where direct measurement is difficult or impossible.

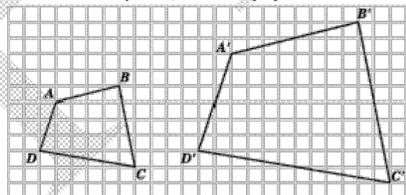
**Note:** The equality of ratios created by similar triangles is of crucial importance in the following study of the slope of a line.

## 7.7 Extend similarity to other polygons in the plane

7.7a Understand both informal and formal definitions of similarity for polygons in the plane.

- Informally, similar polygons are those that have the same shape.
- Formally, two polygons are similar if (a) that they have the same number of sides, (b) the measures of corresponding angles are equal, and (c) the ratios of the lengths of corresponding sides are equal.

**Note:** In the diagram quadrilateral ABCD and  $A'B'C'D'$  are similar. This means that  $\angle A = \angle A'$ ,  $\angle B = \angle B'$ ,  $\angle C = \angle C'$ ,  $\angle D = \angle D'$  and that  $AB/A'B' = BC/B'C' = CD/C'D' = DA/D'A'$ .



**Note:** The value of the common ratio of the sides of similar polygons is called the *scale factor*.

- Understand that congruent figures are similar.
  - Note:** By definition, congruent polygons are those that have the same shape and same size. So they satisfy the "same shape" criterion for similarity.
- Use examples to show that analogues of the SSS, SAS, and AA criteria for similarity of triangles do not work for polygons with more than three sides.

7.7b Understand and use the scale factor through which a polygon can be transformed into a different figure that is similar to the original.

- Relate the scale factor to the scale on maps and in scale model drawings.

**Note:** In Grade 8 similarity will be defined precisely for all figures by means of dilations.

## 7.8 Understand the definition of slope, how to calculate it, and use slope to solve problems.

7.8a Understand and calculate the slope of a line in a coordinate plane.

- The slope of a line is the ratio of the lengths of vertical and horizontal line segments--the "rise" and "run"--that together with the line form a right triangle in a coordinate plane.
- Calculate the slope a line in a coordinate plane.

7.8b Understand the relation of slope to parallel and perpendicular lines in a coordinate plane.

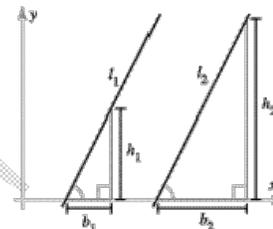
- The calculated slope of a line in a coordinate plane is the same no matter

which two points one uses to perform the calculation.

**Note:** This can be verified using similar triangles.

- Two lines in a coordinate plane are parallel if and only if they have the same slope.

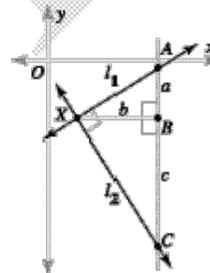
**Note:** Suppose that two lines  $l_1$  and  $l_2$  are parallel (as in the diagram). Then by the corresponding angles property, the angles between these lines and the  $x$ -axis are equal. Since each is a right triangle, it follows from the AA criterion that the two triangles are similar. Therefore,  $h_1/h_2 = b_1/b_2$  and consequently,  $h_1/b_1 = h_2/b_2$ . But the slope of  $l_1 = h_1/b_1$  and the slope of  $l_2 = h_2/b_2$ . Therefore the slope of  $l_1$  equals the slope of  $l_2$ .



To demonstrate the converse, suppose lines  $l_1$  and  $l_2$  have the same slope. Then  $h_1/b_1 = h_2/b_2$ , so the triangles are similar by the SAS criterion. It follows that the angles between the lines and the  $x$ -axis are equal. By the corresponding angles property, the lines must be parallel.

- Two lines in a coordinate plane are perpendicular if and only if the product of their slopes is  $-1$ .

**Note:** Suppose that lines  $l_1$  and  $l_2$  are perpendicular and that the slope of  $l_1$  is  $a/b$  and the slope of  $l_2$  is  $-c/b$ . Triangles  $XAB$  and  $CAX$  are right triangles and share angle  $A$ . Therefore  $\angle AXB = \angle XCB$ , so triangle  $XAB$  is similar to triangle  $XBC$  by the AA criterion for similarity. It follows that  $a/b = b/c$ , and therefore  $(a/b) \times (-c/b) = -1$ .



- 7.8c Find the slopes of physical objects (roads, roofs, ramps, stairs) and express the answers as a decimal, ratio, or percent.

## Geometry

### 8.4 Understand and use the Pythagorean theorem .

8.4a Prove the Pythagorean Theorem and its converse.

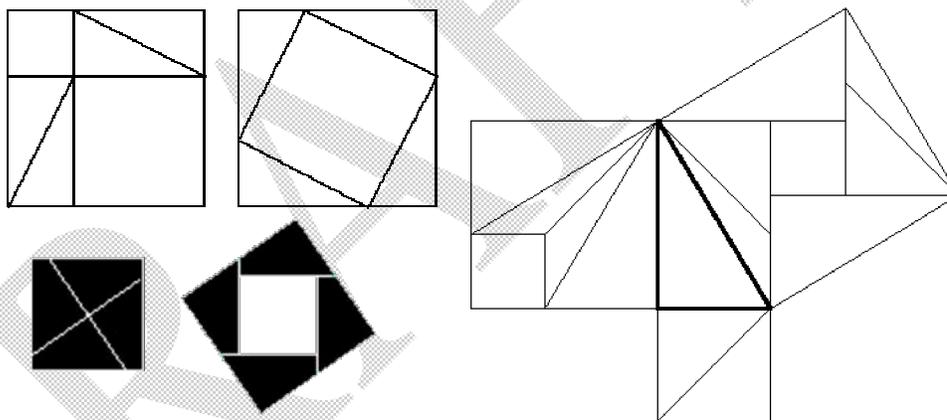
- A suggested proof of the Pythagorean Theorem using simple algebra is shown on the right.
- Other proofs using geometric dissection are shown below.

**Note:** The validity of these proofs depends on the fundamental fact that the sum of the angles in a triangle is  $180^\circ$ .

Area of entire square =  $c^2$

$c^2 = 4\left(\frac{1}{2}ab\right) + (b-a)^2$   
 $c^2 = 2ab + b^2 - 2ab + a^2$   
 $c^2 = b^2 + a^2$   
 $c^2 = a^2 + b^2$

$\frac{1}{2}ab$   
 $\frac{1}{2}ab$   
 $\frac{1}{2}ab$   
 $\frac{1}{2}ab$   
 $(b-a)^2$



8.4b Use the Pythagorean theorem and its converse to find distances between points in the Cartesian coordinate system to solve perimeter, area, and volume problems.

- To make use of the Pythagorean theorem, it is often helpful to draw new lines in a figure in order to create right triangles.

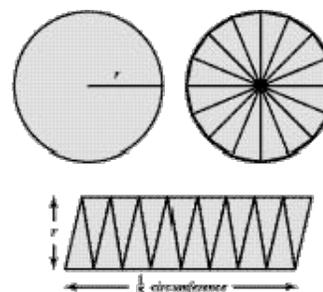
**Example:** Use the fact that the bisector of the angle between the equal sides of an isosceles triangle is the perpendicular bisector of the opposite side.

### 8.5 Understand fundamental properties of a circle.

8.5a Understand and explain relationships among the radius, diameter, circumference, and area of a circle.

- The circumference of a circle is directly proportional to its radius, and to its diameter.

**Note:** The circumference  $C$  of a circle can be thought of intuitively as the limit as the number of sides increases of the perimeters of an inscribed regular polygon. These perimeters are formed from



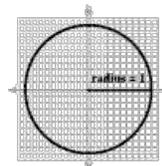
the bases of isosceles triangles whose two equal sides are radii of the circle. By similarity, if the radius of a circle increases, the bases of these triangles will increase in direct proportion.

- The area of a circle equals its radius times one-half of its circumference.  
**Note:** The area  $A$  of a circle can be thought of intuitively as the limit as the number of sides increases of the area of an inscribed regular polygon. The triangular pieces that make up one of these many-sided polygons can be rearranged alternately to form a rectangular strip whose height is approximately the radius  $r$  and whose base is approximately half the circumference  $C$ . Thus  $A = r(C/2)$ .
- The area of a circle is directly proportional the square of its radius (and also to the square of its diameter).  
**Note:** This follows directly from the two preceding statements.  $C = kr$ ;  $A = rC/2$ ; thus  $A = kr^2/2$ .

8.5b The ratio of the circumference to the diameter of a circle is the same as the ratio of the area to the square of the radius. This ratio is called pi, or  $\pi$ .

**Note:** Writing  $C = kd = 2kr$ , we see that  $A = r(C/2) = kr^2$ . Thus  $C:d = A:r^2$ .

- It follows that the area of a unit circle (one whose radius is 1) is  $\pi$ .  
**Note:** Thus one way to approximate  $\pi$  is to use graph paper to approximate the area of a unit circle.
- The value of  $\pi$  is approximately 3.14, or  $22/7$ , or  $3\frac{1}{7}$ .
- Recognize the  $\pi$  is irrational.



8.5c Recognize that a triangle inscribed on the diameter of a circle is a right triangle.

8.5d Recognize that a tangent to a circle forms a right angle with the diameter at the point of tangency.

8.5e Know and use formulas for the circumference and area of a circle, semicircle, and quarter-circle.

8.5g Understand the definition of a great circle on a sphere.

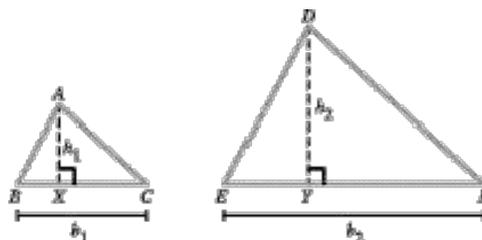
- Recognize that great circles provide shortest routes between points on the surface of a globe (or the earth).

## 8.6 Understand properties of scaling, dilation, and their relation to similarity.

8.6a Understand that similar polygons with scale factor  $r$  have areas related by a factor of  $r^2$ .

- Saying that two triangles are similar with *scale factor*  $r$  means that the ratio of corresponding sides is  $r$ .

**Note:** In triangles similar triangles  $ABC$  and  $DEF$ , the ratios of bases  $b_1/b_2$  and of heights  $h_2/h_1$  equal the scale factor  $r$ . (Indeed, the ratios of all corresponding linear dimensions equal the scale factor  $r$ .) Thus the ratio of the areas of triangles  $DEF$  and  $ABC$  equals  $b_2h_2/b_1h_1 = r^2$ .



- Since any polygon can be decomposed into a finite number of triangles, it follows that any two similar polygons with scale factor  $r$  have areas related by a factor of  $r^2$ .
- By extension, volume expands as the cube of the scale factor.

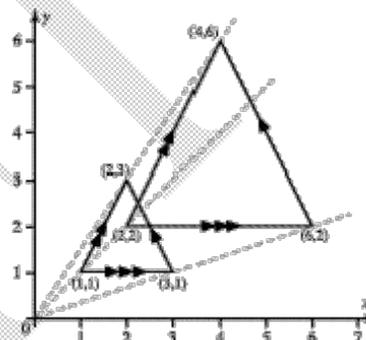
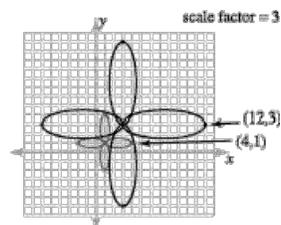
8.6b Understand the definition and properties of dilation, and use it to define similarity of general figures in the plane.

- Informally, dilations are projections of figures in the plane from the perspective of the origin of a coordinate system.
- By definition, a *dilation* centered at the origin with scale factor  $r$  maps the point  $(x, y)$  to the point  $(rx, ry)$ .
- A dilation maps a line to a line with the same slope

**Note:** The slope of a line determined by points  $(x_1, y_1)$  and  $(x_2, y_2)$  equals  $(y_2 - y_1)/(x_2 - x_1)$ . The slope of a line through the points  $(rx_1, ry_1)$  and  $(rx_2, ry_2)$  created by a dilation of scale factor  $r$  equals  $(ry_2 - ry_1)/(rx_2 - rx_1) = (y_2 - y_1)/(x_2 - x_1)$ .

- Dilations map parallel lines to parallel lines (except for those passing through the origin, which do not change).
- **Note:** Lines with the same slope are parallel (or concurrent).
- A dilation maps a triangle to a similar triangle.
- If two triangles in a coordinate system are similar, then there is a dilation moving one of them to a triangle congruent to the other.
- The concept of dilation can be used to define the similarity of arbitrary plane figures.

**Note:** Previously, similarity was defined in terms of angles and ratios of line segments, limiting its applicability to polygonal shapes. Using dilation, similarity can be thought of as "projection from a point." This defines similarity for planar figures of any shape. (For computational simplicity, we use the origin as the projection point, but any other point would do.)



## 8.7 Solve problems involving perimeter, area, volume and surface area.

8.7a Find the perimeter or area of regions in the plane bounded by line segments and circular arcs.

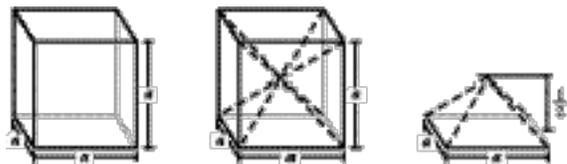
- Use decomposition and triangulation to break a problem into manageable pieces, and use a calculator as necessary for calculations.
- **Note:** To maintain both accuracy and understanding, maintain exact formal calculations until the last step when a calculator should be used to determine (or check) a decimal answer, which in many cases may be just an approximation.

8.7b Understand relationships among volumes of common solids.

- Recognize the 3:2:1 relationship between the volumes of circular cylinders, hemispheres, and cones of the same height and circular base.
- **Note:** These relationships can be effectively demonstrated by pouring water or sand into properly sized plastic molds. They can also be calculated from the formulas  $V = Ah$ ,  $V = (2/3)Ah$ , and  $V = (1/3)Ah$  for the volumes of a right circular cylinder, hemisphere, and cone of height  $h$  and base area  $A$ .

- Recognize that the volume of a pyramid is one-sixth of the volume of the solid that can be decomposed into six identical pyramids.

**Note:** Consider the special case of cube of side  $a$  that has been decomposed into 6 congruent pyramids. The height of each pyramid is  $a/2$ , the area of the base of each pyramid is  $a^2$ , and the volume of each pyramid is one-sixth the volume of the cube. Hence the volume is  $a^3/6 = (1/3) a^2 \cdot (a/2) = (1/3)Ah$  where  $A$  is the area of the base and  $h$  is the height.



- Solve problems involving volumes and surface areas of common solids, including upright cylinders, spheres, cones, pyramids, and rectangular prisms.



## MAP Algebra Expectations

### Algebra

#### K.9 Identify, sort and classify objects.

K.9a Sort and classify objects by attribute and identify objects that do not belong in a particular group.

- Recognize attributes that involve colors, shapes (e.g., triangles, squares, rectangles, and circles), and patterns (e.g., repeated pairs, bilateral symmetry).

**Example:** Identify the common attribute of square in a square book, square table, and square window.

**Example:** Distinguish different patterns in ABABABA, ◆◆◆◆◆◆◆◆◆◆.

K.9c Recognize related addition and subtraction facts.

- Use objects to demonstrate "related facts" such as  $7 - 4 = 3$ ,  $3 + 4 = 7$ ,  $7 - 3 = 4$ .

## Algebra

### 1.8 Recognize and extend simple patterns

- 1.8a Skip count by 2s and 5s, and count backwards from 10.
- 1.8b Identify and explain simple repeating patterns
- Find repeating patterns in the discrete number line, in the 12 x 12 addition table, and in the hundreds table (a 10 x 10 square with numbers arranged from 1 to 100).
    - Note:** Use examples based on linear growth (e.g., height, age).
  - Create and observe numerical patterns on a calculator by repeatedly adding or subtracting the same number from some starting number.
- 1.8c Determine a plausible next term in a given sequence, and give a reason.
- Note:** Without explicit rules, many answers to "next term" problems may be reasonable. So whenever possible, rules for determining the next term should be accurately described. Patterns drawn from number and geometry generally have clear rules; patterns observed in collected data generally do not.

### 1.9 Find unknowns in problems involving addition and subtraction.

- 1.9a Understand that addition can be done in any order but that subtraction can not.
- Demonstrate using objects that the order in which things are added does not change the total, but that the order in which things are subtracted does matter.
  - Use the fact that  $a + b = b + a$  to simplify addition problems.
    - Examples:**  $2 + 13 = 13 + 2 = 15$  (by adding on);  
 $7 + 8 + 3 = 7 + 3 + 8 = 10 + 8 = 18$ .
    - Note:** The relation  $a + b = b + a$  is known as the *commutative* property of addition. It reduces significantly the number of addition facts that need to be learned. However, the vocabulary is not needed until later grades.
  - Demonstrate understanding of the basic formula  $a + b = c$  by using objects to illustrate all eight number sentences associated with any particular sum:
    - Example:**  $8 + 6 = 14$ ,  $6 + 8 = 14$ ;  $14 = 8 + 6$ ,  $14 = 6 + 8$ ;  
 $14 - 8 = 6$ ,  $6 = 14 - 8$ ;  $14 - 6 = 8$ ,  $8 = 14 - 6$ .

### 1.10 Understand how adding and subtracting are inverse operations.

- 1.10a Demonstrate using objects that subtraction undoes addition, and *vice versa*.
- Subtracting a number undoes the effect of adding that number, thus restoring the original. Similarly, adding a number undoes the action of subtracting that number.
    - Example:**  $2 + 3 = 5$  implies  $5 - 2 = 3$  and  $5 - 2 = 3$  implies  $2 + 3 = 5$ .
  - Use the inverse relation between addition and subtraction to check arithmetic calculations.
    - Note:** Addition and subtraction are said to be *inverse* operations because subtraction undoes addition and addition undoes subtraction. However, this vocabulary is not needed until later grades.
    - Caution:** Subtraction is sometimes said to be equivalent to "adding the opposite," meaning that  $5 - 3$  is the same as  $5 + -3$ . Here the "opposite" of a number is intended to mean the negative of a number. However, since negative numbers are not introduced until later grades, this formulation of the relation between addition and subtraction should be postponed.

## Algebra

### 2.11 Create, identify, describe, and extend patterns.

2.11a Fill in tables based on stated rules to reveal patterns.

- Find patterns in both arithmetic and geometric contexts.

2.11b Record and study patterns in lists of numbers created by repeated addition or subtraction.

- Create patterns mentally (by counting up and down), by hand (with paper and pencil), and by repeated action on a calculator.

**Examples:** 3, 8, 13, 18, 23, ...; 50, 46, 42, 38, 34, ...

### 2.12 Find unknowns in simple arithmetic problems

2.12a Solve equations and problems involving addition, subtraction and multiplication with the unknown in any position.

**Note:** In the early grades it is better to signify the unknown with a symbol such as [ ], ?, or □ that carries the connotation of unknown rather than with an alphabetic letter such as x.

2.12b Understand and use the facts that addition and multiplication are commutative and associative.

- Use parentheses to clarify groupings and order of operation.
- Recognize terms such as *commutative* and *associative*.

**Note:** It is not necessary for children at this grade to use or write these words, merely to recognize them orally and to know the properties to which they refer.

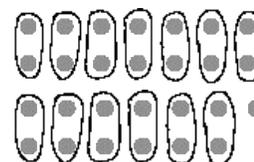
2.12c Recognize how multiplication and division are, like addition and subtraction, inverse operations.

### 2.13 Understand basic properties of odd and even numbers.

2.13a Explain why the sum of two even numbers is even, and that the sum of two odd numbers is also even.

- Use diagrams to represent even and odd numbers and to explain their behavior.

**Example:** The representation at the right shows that 14 is even and 13 is odd.



2.13b Answer similar questions about subtraction and multiplication of odd or even numbers.

## Algebra

### 3.12 Explore and understand arithmetic relationships among positive whole numbers.

3.12a Understand the inverse relationships between addition and subtraction and between multiplication and division, and the commutative laws of multiplication and addition.

- Show that subtraction and division is not commutative.

3.12b Find the unknown in simple equations that involve one or more of the four arithmetic operations.

**Note:** To emphasize the process of solving for an unknown, limit coefficients and solutions to small positive whole numbers.

**Examples:**  $3 \times ? = 3 + 6$ ;  $? \div 5 = 5 \times 55$ ;  $36 = ? \times ?$ .

3.12c Create, describe, explain, and extend patterns based on numbers, operations, geometric objects, and relationships.

- Explore both arithmetic (constant difference) and geometric (constant multiple) sequences.

**Examples:** 100, 93, 86, 79, 72, ...; 2, 4, 8, 16, ...; 3, 9, 27, 81, ... .

- Understand that patterns do not imply rules; rules imply patterns.

## Algebra

### 4.12 Use properties of arithmetic to solve simple problems.

4.12a Understand and use the commutative, associative, and distributive properties of numbers.

- Use these terms appropriately in oral descriptions of mathematical reasoning.
- Use parentheses to illustrate and clarify these properties.

4.12b Find the unknown in simple linear equations.

- Use a mixture of whole numbers, fractions, and mixed numbers as coefficients.

**Examples:**  $24 + n = n - 2$ ;  $3/4 + p = 5/4 - p$

**Note:** "Simple" equations for Grade 4 are those that require only addition or subtraction (e.g.,  $3/4 + [ ] = 7/4$ ) or a single division whose answer is a whole number (e.g.,  $3 \times [ ] = 12$ ).

**Note:** There is no need to use the term *linear* since these are the only kinds of equations encountered in Grade 4.

### 4.13 Evaluate simple expressions

4.13a Find the value of expressions such as  $na + b$  and  $na - b$  where  $a$ ,  $b$ , and  $n$  are whole numbers or fractions and where  $na \geq b$ .

- Make tables and graphs to display the results of evaluating expressions for different values of  $n$  such as  $n = 1, 2, 3, \dots$ .

**Note:** Evaluating an expression involves two distinct steps: substituting specific values for letter variables in the expression, and then carrying out the arithmetic operations implied by the expression. Working with expressions both introduces the processes of algebra and also reinforces skills in arithmetic.

**Note:** Avoid negative numbers since systematic treatment of operations on negative numbers is not introduced until grade 6.

4.13b Evaluate expressions such as  $\frac{a}{b} + \frac{c}{nb}$ , where  $a$ ,  $b$ ,  $c$ , and  $n$  are whole numbers.

4.13c Evaluate expressions such as  $\frac{1}{a} + \frac{1}{b}$  where  $a$  and  $b$  are single digit whole numbers.

**Example:** The value of  $\frac{a}{b} + \frac{c}{nb}$  when  $a = 1$ ,  $b = 2$ ,  $c = 3$ , and  $n = 4$  is

$$\frac{1}{2} + \frac{3}{8} = \frac{4}{8} + \frac{3}{8} = \frac{7}{8}.$$

**Note:** Addition of fractions is limited to cases included in the Grade 4 expectations, namely, unit fractions with denominators under 10 and other fractions where one denominator is a multiple of the other.

## Algebra

### 5.16 Find the unknown in simple linear equations.

5.16a Equations that require only simple calculation should be solved mentally (that is, "by inspection"):

$$96 + 67 = b + 67$$

$$\frac{3}{4} + \frac{5}{8} - \frac{5}{8} = p$$

$$a + \frac{3}{5} = \frac{3}{5}$$

$$39 - k = 39 - 40$$

$$\frac{3}{5} - \frac{3}{8} + \frac{5}{8} = d + \frac{3}{8} - \frac{5}{8}$$

$$\frac{1}{5} + \frac{2}{5} = b + \frac{6}{5}$$

$$78 + b = 57 + 79$$

$$53 + 76 = 51 + 76 + d$$

### 5.17 Evaluate and represent simple expressions.

5.17a Translate between simple expressions, tables of data, and graphs in the coordinate plane.

5.17b Understand and use the conventions for order of operations (including powers).

**Example:**  $ax^2 + bx = (a(x^2)) + (bx)$ , not  $(ax)^2 + bx$ .

5.17c Evaluate expressions such as

- $nr$  where  $n$  is a whole number and  $r$  is a fraction.
- $nab/(na-b)$  when  $n$ ,  $a$  and  $b$  are whole numbers and where  $na > b$ .
- $\frac{a}{b} + \frac{c}{d}$ , where  $a$ ,  $b$ ,  $c$ ,  $d$  are positive whole numbers.
- $1/ab$  where  $a$  and  $b$  are positive whole numbers.
- $a/b$  where one of  $a$  or  $b$  is a positive whole number and one is a fraction.

**Note:** Avoid expressions that introduce negative numbers since systematic treatment of operations on negative numbers is not introduced until grade 6.

**Note:** Working with expressions both introduces the processes of algebra and also reinforces skills in arithmetic.

5.17d Understand the importance of not dividing by zero.

## Algebra

### 6.11 Understand that the system of negative and positive numbers obeys and extends the laws governing positive numbers.

6.11a The sum and product of two numbers, whether positive or negative, integer or fraction, satisfies the commutative, associative and distributive laws.

- For any numbers  $a, b, c$ , (whether positive, negative, or zero),  
 $a + b = b + a$ ,  $a \times b = b \times a$  (commutative);  
 $a + (b + c) = (a + b) + c$ ;  $a \times (b \times c) = (a \times b) \times c$  (associative);  
 $a \times (b + c) = (a \times b) + (a \times c)$  (distributive).

**Example:**  $(-3) \times 5 = -(3 \times 5)$ , because  $((-3) \times 5) + (3 \times 5) = ((-3) + 3) \times 5 = 0 \times 5 = 0$ . Hence the sum of  $(-3) \times 5$  and  $3 \times 5$  is 0, so  $(-3) \times 5 = -(3 \times 5)$ .

**Note:** This example can usefully be demonstrated on the number line in a way that avoids the formality of parentheses required above.

6.11b Understand why the product of two negative numbers must be positive.

- Since a negative number  $-a$  is defined by the equation  $-a + a = 0$ , the distributive law forces the product of two negative numbers to be positive.

**Example:** To show that  $(-3) \times (-5) = 3 \times 5$ , we demonstrate that the sum of the left side  $[(-3) \times (-5)]$  with the negative of the right  $[3 \times 5]$  is zero:

$$\begin{aligned} ((-3) \times (-5)) + (-3 \times 5) &= ((-3) \times (-5)) + ((-3) \times 5) \\ &= (-3) \times ((-5) + 5) = (-3) \times 0 = 0. \end{aligned}$$

The key middle step uses the distributive law.

6.11c Understand why the quotient of two negative numbers must be positive.

- Division is the same as multiplication by a reciprocal. If a number  $b$  is negative, so is its reciprocal  $1/b$ . So if  $a$  and  $b$  are both negative,  $a/b$  is positive since it equals the product of two negative numbers:  $a \times (1/b)$ .

**Example:** If  $p = -12/-3$ , then  $-12 = p \times (-3)$ . Since  $4 \times -3 = -12$ , this yields  $p = 4$ , hence  $-12/-3 = 12/3 = 4$ .

### 6.12 Represent and use algebraic relationships in a variety of ways.

6.12a Recognize and observe notational conventions in algebraic expressions.

- Understand and use letters to stand for numbers.
- Recognize the use of juxtaposition (e.g.,  $3x$ ,  $ab$ ) to stand for multiplication, and the convention in these cases of writing numbers before letters.
- Recognize the tradition of using certain letters in particular contexts.  
**Note:** Most common:  $k$  for constant,  $n$  for whole number,  $t$  for time, early letters ( $a, b, c$ ) for parameters, late letters ( $u, v, x, y, z$ ) for unknowns.
- Recognize different conventions used in calculator and computer spreadsheets (e.g.,  $*$  for multiplication,  $^$  for power).
- Understand and use conventions concerning order of operations and use parentheses to specify order when necessary.  
**Note:** By convention, powers are calculated before multiplication (or division), and multiplication is done before addition (or subtraction).  
**Example:**  $3x^2$  means  $3(x \cdot x)$ , not  $(3x) \cdot (3x)$ ;  $3x^2 - 7x$  means  $(3x^2) - (7x)$ , not  $(3x^2 - 7) \cdot x$ .
- Evaluate expressions involving all five arithmetic operations (addition, subtraction, multiplication, division, and power).

**Note:** For the most part this is review. What is new is the introduction of powers and the shift to use of letters (rather than boxes, question marks and other placeholders) to represent generic or unknown numbers.

**Examples:**  $3x^2 - 7x + 2$ , when  $x = 3$  or  $1/3$ ;  $2x^3 + x$ , when  $x = 2$ , or  $1/2$ , or  $0$ , or  $-3$ ;  $3x^2 - 2xy$ , when  $x = 2$  and  $y = 6$ .

6.12b Solve problems involving translation among and between verbal, tabular, numerical, algebraic, and graphical expressions.

- Write an equation that generates a given table of values

**Note:** Limit examples to linear relationships with integer domains.

- Graph ordered pairs of integers on a coordinate grid.

**Example:** Prepare *scatterplots* of related data such as students' height vs arm length in inches.

- Generate data and graph relationships concerning measurement of length, area, volume, weight, time, temperature, money, and information.
- Understand why formulas or words can *represent* relationships whereas tables and graphs can generally only *suggest* relationships.

**Note:** Unless the rule for a table (or graph) is specified (e.g., in a tax table), the numbers themselves cannot determine a relationship that extends to numbers not in the table.

**Note:** The issue here is about the lack of precision inherent in a graph, not about the possibility, which is present in some graphs, of ambiguity concerning which of two very different points on the graph are associated with a given input value. The common example of ambiguity concerns the graph of the equation for a circle, but such graphs are not among those studied in grades K-8.

## Algebra

### 7.9 Understand the properties of linear functions and use these properties in a variety of applications.

7.9a Understand the concept of a function as a rule that assigns one number (the output) to another number (the input).

- The notation  $f(x)$  represents the number that the function  $f$  assigns to the input  $x$ .
- Make tables of inputs  $x$  and outputs  $f(x)$  for a variety of rules that take numbers as inputs and produce numbers as outputs.
- Plot points on a coordinate plane to represent tables of function values.
- Understand that the *graph* of a function  $f$  is the set of points in the coordinate plane representing the ordered pairs  $(x, f(x))$ .

**Note:** However, do *not* define a function as a set of ordered pairs.

- Use simple algebraic expressions to define rules for functions to be calculated, tabulated, and graphed.

**Note:** While functions are often defined by formulas, they can be defined by other sorts of rules.

**Note:** A ubiquitous source of function rules can be found on the function feature of spreadsheet software. Most of these functions are too advanced for middle grades, but many are not. They provide good sources of examples.

7.9b A function exhibiting a constant rate of change is called a *linear* function.

- A *constant rate of change* means that when the input changes at a constant rate, so does the output.

**Examples:**  $f(x) = 2x$ ;  $f(x) = 5-3x$ ;  $f(\text{side of square}) = \text{perimeter of square}$ .

7.9c A linear function in which  $f(0) = 0$  is called a *proportional relationship*.

- A proportional relationship can be represented by  $f(x) = kx$  where  $k$  is called the *constant of proportionality*.
- The graph of a proportional relationship  $f(x) = kx$  is a line with slope  $k$  that passes through the point  $(0, 0)$ .

**Note:** Similar triangles show that the point  $(x, kx)$  for any number  $x$  is on the line that connects  $(0,0)$  to  $(1,k)$ .

7.9d A linear function can be represented by the function  $f(x) = mx + b$ .

- If  $f(x) = mx + b$ , then the function  $g(x) = f(x) - b = mx$  is a proportional relationship.

**Note:** If  $f(x)$  is a linear function, then  $g(x) = f(x) - f(0)$  will be a proportional relationship (since  $g(0) = 0$ ). That means that  $g(x) = kx$ , so  $f(x) = g(x) + f(0) = kx + f(0)$ . Setting  $b = f(0)$  yields  $f(x) = kx + b$ .

**Note:** By tradition, the constant of proportionality in a proportional relationship is usually denoted by  $k$ , while the similar constant in a linear function is denoted by the letter  $m$ .

**Note:** The function  $f(x) = mx + b$  is often written as  $y = mx + b$  since when  $f(x)$  is graphed on a coordinate grid the values of  $f(x)$  are marked off on the vertical  $y$ -axis.

7.9e The graph of a linear function  $f(x) = mx + b$  is a straight line.

**Note:** The graph of  $f(x) = mx + b$  is the graph of the proportional relationship  $g(x) = mx$  shifted up (or down) by  $b$  units. Since the graph of  $g(x)$  is a straight line, so is the graph of  $f(x)$ .

- In the graph of  $f(x) = mx + b$ ,  $m$  is the slope of the line and  $b$  is the  $y$ -coordinate of the point where the graph crosses the  $y$ -axis (i.e., the value of  $f(x)$  when  $x = 0$ ).
 

**Note:** Slopes provide an algebraic way to show that the graph of  $f(x) = mx + b$  is a straight line. Consider the slopes determined by any two points on the graph of  $f(x)$ . Since  $f(x_1) = mx_1 + b$  and  $f(x_2) = mx_2 + b$ , it follows that  $f(x_1) - f(x_2) = m(x_1 - x_2)$ . Hence  $m = (f(x_1) - f(x_2)) / (x_1 - x_2)$  is the same for any two points on the graph of  $f(x)$ .

**Example:** Compare and contrast the graphs of  $x = k$ ,  $y = k$ , and  $y = kx$ , where  $k$  is a constant.

7.9f Translate common linear phenomena into symbolic statements and interpret  $m$  and  $b$  in  $f(x) = mx + b$  in terms of the original situation.

- Work fluently with directly proportional relationships and linear functions.
- Recognize contextual situations in which linear models are appropriate.
- Solve problems involving linear phenomena.
 

**Note:** Common examples of linear phenomena include distance and time under constant speed; shipping costs under constant incremental cost per pound; conversion of measurement units (pounds and kilograms, degrees Celsius and degrees Fahrenheit); cost of gas in relation to gallons used; the height and weight of a stack of chairs.

## 7.10 Plot and interpret graphs of functions representing different non-linear relationships.

7.10a Create tables of coordinate points and plot graphs of various functions that are not linear.

- Plot points of non-linear functions by hand to gain experience with the coordinate system.
 

**Note:** To understand what kinds of functions are linear it is important for students to work with examples of functions that are not linear.

**Examples:**  $f(x) = 3x^2 + 1$ ,  $f(x) = 2x^3$ , and  $f(x) = 5/x$ .

7.10b An *inversely proportional* relationship is represented by  $f(x) = k/x$  where  $k$  is some non-zero number.

- The graph of  $f(x) = k/x$  is a curve, not a line, and does not cross either the  $x$  nor the  $y$ -axis.
 

**Note:** There is no value of  $x$  for which  $f(x) = 0$ , nor is there any value for  $f(x)$  if  $x = 0$ .

**Note:** To more sharply contrast a proportional relationship from an inversely proportional relationship, the former is often called a *directly proportional* relationship.
- Recognize quantities that are inversely proportional, for example, the relationship between lengths of the base and side of a rectangle with fixed area.

7.10c A *quadratic* function is represented by an expression of the form  $f(x) = ax^2 + bx + c$  where  $a$ ,  $b$ , and  $c$  are specific numbers, and  $x$  represents the input into the function.

- Except when  $a = 0$ , the graph of  $f(x) = ax^2 + bx + c$  is a curve that always crosses the  $y$  axis but may or may not cross the  $x$ -axis.
 

**Note:** When  $x = 0$ ,  $f(x) = c$ , so the graph of  $f(x)$  crosses the  $y$ -axis at the point  $(0, c)$ . When  $a = 0$ ,  $f(x) = bx + c$  which is a linear function.

- Quadratic functions may also appear in *factored form*:  $f(x) = (x - r)(x - s)$ .  
**Note:** In this form,  $f(x) = 0$  whenever  $x = r$  or  $x = s$ . So the graph of the function  $f(x)$  crosses the  $x$  axis at these two points.
- Recognize quantities that are represented by quadratic functions, for example, the relationship between lengths of the side of a square and its area.

7.10d Recognize whether information given in a table, graph, or formula suggests a direct proportional, linear, quadratic, inversely proportional, or other nonlinear relationship.

- Be able to identify graphs of simple linear and quadratic functions.

7.10e Solve simple problems involving relationships that are directly proportional, linear, quadratic, or inversely proportional.

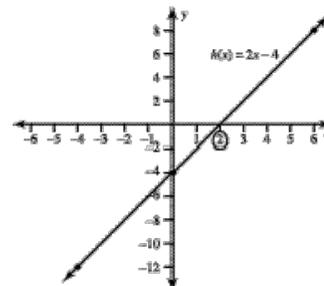
## 7.11 Understand the concept of equation and the relation among equations, functions, and graphs.

7.11a Solve linear equations by graphing.

- If  $f(x)$  and  $g(x)$  are functions, the expression  $f(x) = g(x)$  is called an *equation*. *Solving an equation* means finding all value of  $x$  for which the equation is true.
- A common special case is when  $g(x) = 0$ . In this case solving the equation  $f(x) = 0$  means finding all value of  $x$  for which  $f(x) = 0$ .
- The solution to the equation  $f(x) = 0$  are the values of  $x$  where the graph of the function  $f(x)$  crosses the  $x$  axis.

**Example.** As indicated by the illustration on the right, the graph of the linear function  $f(x) = 2x - 4$  crosses the  $x$ -axis at  $x = 2$ . This is the solution to the equation  $f(x) = 0$ , since  $f(2) = 2 \cdot 2 - 4 = 0$ .

**Note:** Since in the coordinate plane the values of  $f(x)$  are plotted in the vertical direction along the  $y$ -axis, expressions such as  $y = 7x + 5$  or  $y = 3x^2 - 15x$  are often called equations. Graphing such an equation means graphing the function  $f(x) = 7x + 5$  or  $f(x) = 3x^2 - 15x$ . Solving such an equation means finding the value of  $x$  for which  $y$ , that is,  $f(x)$  equals 0.



7.11b Solve linear equations by algebraic simplification.

- Use established properties of numbers to simplify expressions associated with linear functions.
- Add, subtract, and multiply simple polynomial expressions.

**Example:**  $(2x+5) + (3-2x) = 2x + 5 + 3 - 2x = 8$   
 $(2x+5) - (3-2x) = 2x + 5 - 3 + 2x = 2 + 4x$   
 $(2x+5) \times (3-2x) = 6x + 15 - 4x^2 - 10x = 15 - 4x - 4x^2$

- Recognize and use the fact that equals added to equals are equal and that equals multiplied by equals are equal.

**Note:** More formally, if  $A = B$  and  $C = D$ , then  $A+B = C+D$  and  $AC=BD$ . These rules apply to polynomial expressions since each such expression represents a number, and the rules apply to numbers.

**Caution:** Be alert to anomalies caused by dividing by 0 (which is, of course, undefined), or by multiplying both sides by 0 (which will produce equality even when things were originally unequal). For example, multiplying both sides of an equation by  $x-1$  is appropriate only when  $x \neq 1$ .

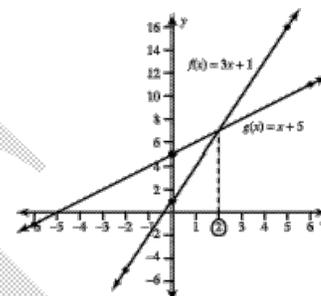
## Algebra

### 8.8 Understand and solve problems involving linear equations in one and two variables.

- 8.8a Understand why the solution to the linear equation  $f(x) = g(x)$  is the x-value of the intersection of the graphs of  $f(x)$  and  $g(x)$ .

**Example:** To solve the linear equation  $3x + 1 = x + 5$ , graph  $f(x) = 3x + 1$  and  $g(x) = x + 5$ . The intersection of the graphs of  $f(x)$  and  $g(x)$  shows that the solution to the linear equation is  $x = 2$ .

**Note:** Alternatively, the linear equation  $3x + 1 = x + 5$  can be rewritten as  $2x - 4 = 0$ . This yields a single linear function  $h(x) = 2x - 4$  which can be solved by graphing  $h(x)$  (see end of Grade7).



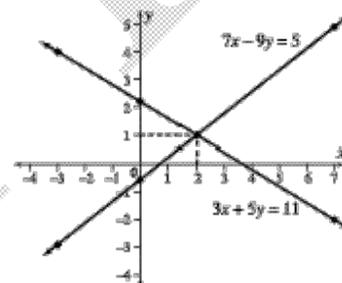
- 8.8b Solve simultaneous linear equations in two variables by graphing.

- A linear equation *in two variables* is an expression of the form  $ax + by = c$ . The *graph* of such an equation consists of all points  $(x, y)$  in the coordinate plane that satisfy the equation.

**Note:** Graphs of linear equation in two variables are straight lines.

- A system of *simultaneous linear equations in two variables* consists of two different linear equations in two variables. A *solution* to such a system consists of specific values  $x_0, y_0$  that make both equations true.
- The coordinates of the point of intersection of the graphs of two linear equations are the solutions to the system of equations.

**Example:** To solve the system  $3x + 5y = 11$ ;  $7x - 9y = 5$ , first graph each of the two equations (see figure on the right). The point  $x_0 = 2$  and  $y_0 = 1$  where the graphs of the two equations intersect lies on both graphs, so it is a solution of both equations.



- 8.8c Solve simultaneous linear equations in two variables by substitution.

### 8.9 Understand and solve problems involving linear inequalities in one and two variables.

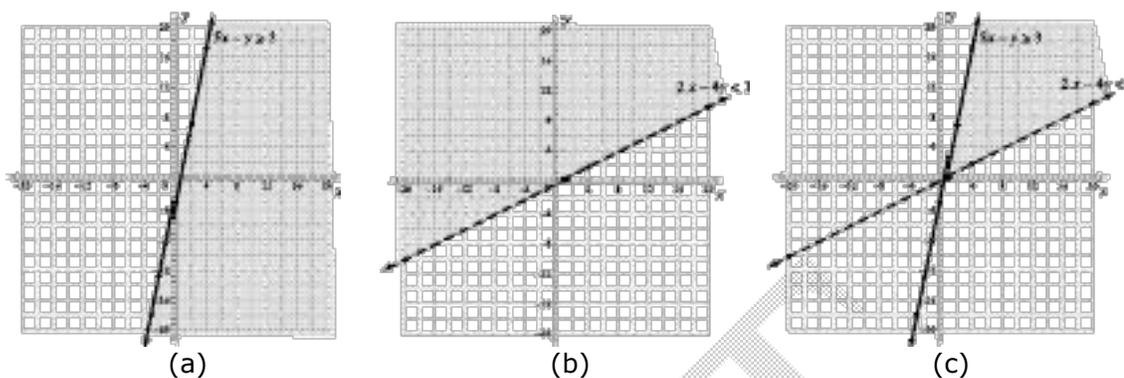
- 8.9a Understand why the graph of a linear inequality is a half plane.

- In analogy with the vocabulary of equations, the collection of all points  $(x, y)$  that satisfy the *linear inequality*  $ax + by \leq c$  is called the *graph* of the inequality. These points lie entirely on one side of the line that is the graph of the equation  $ax + by = c$ .

- 8.9b Understand that when both sides of an inequality are multiplied or divided by a negative number the inequality is reversed, but that all other basic operations when applied to both sides preserve the inequality.

- 8.9c Solve linear inequalities in one and two variables.

**Examples:** Graphs (a) and (b) illustrate that the graph of a linear inequality is a half plane. Graph (c) illustrates a solution to the question: What is the set of points  $(x, y)$  that satisfies both  $5x - y \geq 3$  and  $2x - 4y < 1$ ?



## 8.10 Know, understand, and use common nonlinear functions, identities, and equations.

8.10a Make regular fluent use of basic algebraic identities.

- These include:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

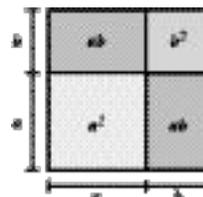
$$(a + b)(a - b) = a^2 - b^2$$

**Example:**  $37 \times 43 = (40-3)(40+3) = 1400-9 = 1391$ .

- Use the distributive law to derive each of these formulas.

**Note:** These identities are used in algebraic proofs of the Pythagorean theorem.

- Understand how the partitioned square on the right provides a geometric representation of  $(a + b)^2 = a^2 + 2ab + b^2$ .



8.10b Recognize common nonlinear functions and their graphs.

- These include:
  - inverse functions such as  $f(x) = k/(bx)$ ;
  - quadratic functions such as  $f(x) = ax^2 + b$  or  $g(x) = (x-a)(x+b)$ ;
  - cubic functions, such as  $f(x) = x^3$  and  $f(x) = x^3 - a$ ;
  - square root functions, such as  $f(x) = 5\sqrt{x}$ ;
  - exponential functions with an integer base, such as  $f(n) = 2^n$  for positive integers  $n$ .
- Decide if a given graph or table of values suggests an inverse, quadratic, cubic, square root, or exponential function.

8.10c Work fluently with nonlinear functions in geometric contexts of length, area, and volume.

- Examples include:
  - the area and radius of a circle;
  - the volume and radius of a sphere;
  - the number of diagonals and the number of sides of a polygon;
  - the areas of simple plane figures and their linear dimensions;
  - the surface areas and volumes of simple three-dimensional solids and their linear dimensions.

8.10d Graph quadratic functions and use the graph to locate roots.

- A *root* of a function  $f(x)$  is a value of  $x$  for which  $f(x) = 0$ .

**Note:** Beware the confusion inherent in two apparently different meanings of root: the *root of a function* (e.g.,  $f(x) = 3x^2 - 4x + 1$ ) and the *root of a number* (e.g.,  $\sqrt{5}$ ). Although different, these uses do arise from a common source: the root of a number such as  $\sqrt{5}$  is the root of an associated equation  $f(x) = x^2 - 5$ .

- Make tables of values of quadratic functions and plot points by hand.  
**Note:** To gain experience, students need to do some graphing by hand rather than by using a graphing calculator. However, a calculator can be helpful to fill out a table of values for more complicated functions.
- For quadratic functions that are given in factored form  $f(x) = (x - a)(x - b)$ , use the roots of the equation  $f(x) = 0$  to sketch the graph of  $f(x)$ .  
**Note:** This strategy also works for simple quadratic functions that can be factored using an algebraic identity, e.g.,  $f(x) = x^2 - a^2 = (x+a)(x-a)$ .
- Estimate the roots of a quadratic equation from the graph of the corresponding function.

#### 8.10e Solve factorable quadratic equations.

- Include equations of the form  $(x+a)^2 = b$ .  
**Note:** In Grade 8, problems involving quadratics should be limited to recognizably factorable examples. General methods of solution such as completing the square, the quadratic formula, and calculator approximations should be deferred until a later grade.